REALE ACCADEMIA DELLE SCIENZE DI TORINO

Plana. IV. 26

PRINCIPIORUM

CALCULI DIFFERENTIALIS

ET

INTEGRALIS

EXPOSITIO ELEMENTARIS

AD NORMAM DISSERTATIONIS AB ACADEMIA SCIENT. REG. PRUSSICA ANNO 1786. PRÆMII HONORE DECORATÆ ELABORATA

AUCTORE

SIMONE L'HUILIER

ACADEMIÆ SCIENT. REGIÆ PRUSSICÆ ET SOCIETATIS REGIÆ LONDINENSIS SOCIO ACADEMIÆ IMPERIALIS PETROPOLITANÆ CORRESPONDENTE.

ACCADEMIA
R. DELLE
DI TORINO

L'Infini est le goufre où ſe perdent nos penſees.

BAILLY Hist. de l'Astron. mod.

TUBINGÆ

APUD JOH. GEORG. COTTAM.

1795.

ILLUSTRISSIMO EXCELLENTISSIMO

DOMINO

COMITI AB HERZBERG

REGIÆ SCIENTIARUM ARTIUMQUE LIBERALIUM ACADEMIÆ PRUSSICÆ CURATORI
OMNIUM BONARUM LITERARUM FAUTORI ET ARBITRO PERITISSIMO
STUDIOSISSIMO

VIRIS ILLUSTRIBUS DOCTISSIMIS CELEBERRIMIS
REGIÆ SCIENTIARUM ELEGANTIORUMQUE LITERARUM ACADEMIÆ
PRUSSICÆ MEMBRIS

OPUS HOC

QUOD SUFFRAGIIS IPSORUM EXCITATUS ACCURATIUS ELABORARE
EORUMQUE ATTENTIONE DIGNIUS REDDERE
ANNIXUS EST

R. DELLE SCIENZE
DI TORINO

D. D. D.

AUCTOR.

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INTRO-

INTRODUCTIO.

Quo amplior eſt ſcientiæ alicujus ambitus, & quo uberiores gravioresque ſunt ejus applicationes; eo magis neceſſarium eſt principiis certis ac dilucidis ipſam ſuperſtruere, & conſequentiarum cum principiis nexum ab omni dubio ac difficultate tutum atque immunem præſtare. Priorem prærogativam doctrinis mathematicis competere in confeſſo eſt; ut altera gaudeant, qui eas meditantur atque exponunt, debent curare. Mathematici veteres, methodi rigoroſæ, qua ipſorum opera nitent, ſedulo tenaces, exemplar nobis præbuerunt modi in ſcientiis his ſtabiliendis atque explicandis adhibendi, ut nomen exactarum mereantur. Recentiores, ſtadium ab antiquis panſum ingreſſi, veſtigiis ipſorum non ſemper inſtiterunt; neque progreſſus ſuos omnino ad normam ab illis conſtitutam compoſuerunt. Speciatim veteres quantitatem ſemper, notioni ipſius conformiter, tanquam augmenti & decrementi capacem, proinde ut ineptam recipiendo ultimo cuidam magnitudinis vel parvitatis termino, conſiderarunt. Recentiores contra, quantitatem in utroque hoc ſtatu extremo ratiociniis & calculis ſubjici poſſe rati, quantitates (etiamnum ſic dictas) infinite magnas & infinite parvas adoptarunt; peculiaremque effinxerunt infiniti ſcientiam, cui partes matheſeos fundamentis ſolidioribus nixæ opitulentur quidem, ſed quæ ab illis ſit tam objecto, quam principiis diverſa: quo audaci conatu doctrinas ab antiquis transmiſſas mirifice auxerunt.

Difficultas, vel impoſſibilitas potius ſtatus illos quantitatis (etiamnum ſic dictæ) definiendi impoſſibilitatem prodit exiſtentiæ eorum & conceptus, quo apprehendantur. Non aggrediar hoc loco definitiones, quæ propoſitæ a variis fuerunt, diſcutere. Interim aſſerere auſim, eas vel contradictiones implicare, quæ

quæ dubiis obnoxias reddant conſequentias ex ipſis deductas; vel adeo vagas eſſe & indeterminatas, ut juxta eas nonniſi propemodum ac relate ad propoſitum quendam ſcopum peculiarem vera eſſent, quæ per analyſin, quam vocant, infiniteſimalem eruerentur.

Sententiam hanc de indole ſtatuum illorum quantitatis profiteri mihi viſa eſt illuſtris Academia ſcientiarum Berolinenſis in Programmate, quo theoriam infiniti mathematici præmio ab ſe anno 1786. condecorandam promulgavit. Quare ab ſcopo ipſius alienum haud fore exiſtimavi, ſi matheſin infinito carere poſſe oſtenderem, vaſtiſſimamque ac maxime ſublimem ejus partem ad eadem principia reducerem, quibus veterum inventa nituntur; quorum vero vel opem recentiores plerique juſto vilius penderunt, vel fœcundati nimium diffiſi ſunt.

Evincere igitur inſtitui: methodum exhauſtionis ſeu limitum ab antiquis excultam, & in Euclidis præſertim atque Archimedis, quæ ad nos pervenere, ſcriptis traditam, ſi congruenter extendatur, novorum calculorum principiis ſolide ac dilucide ſtabiliendis ſufficere. Honor, quem illuſtris Academia conatui meo detulit, utilem eum eſſe mihi perſuaſit; atque ad objectum hoc arduum novis curis tractandum, Diſſertationemque & tempore nimis brevi & loco minus commodo conſcriptam perficiendam me exſtimulavit.

Scriptum meum, quod novum appellare poſſe credo, juris publici eo confidentius facio, quod judici non incongruo fuit probatum. Turbas inter ſatis proh dolor notas, quibus patria mea fuit exagitata, tranquillitate animi ad meditandum neceſſaria frui non poteram. Amicus meus, Dn. Pfleiderer, Phys. & Math. Prof. in Univerſitate Tubingenſi, refugium mihi ſecum obtulit: ubi ſtudiis ad levandos dolores intentus, quæ ad novam Diſſertationis meæ editionem, dudum neceſſariam viſam, ſparſim præparaveram, in ordinem redegi, & cum amico communicavi, ipſiusque obſervationes conſului; quo factum eſſe ſpero, ut, quod publico nunc offero, ſcriptum utile ſit nec ejus attentione indignum.

Juxta ejusdem amici per literas antea jam mecum communicatas, nec non Dni. Prévost, Prof. Philos. Genevenſis, in Diſſertationem meam præmio ornatam

natam animadverſiones amplius extendi primum ejus Caput; quod, cum fundamenta exhibeat totius ſyſtematis calculorum ſuperiorum, præcipua cura erat tractandum. Speciatim notio limitis ibidem tradita nimis erat anguſta; &, ut omnes complecteretur caſus, debebat, uti nunc fit §. 13, extendi.

Ita factum eſt, ut Caput hoc fundamentale augmentum ſatis inſigne nanciſceretur; neque rationem habendam eſſe exiſtimavi cenſuræ prolixitatis horum præliminarium. Adeo perſuaſum habeo, principia calculorum ſuperiorum debere ad methodum limitum reduci, nec poſſe alio modo ſolide ſtabiliri; ut neceſſarium eſſe putaverim, methodum illam in formam doctrinæ redigere. Quæ cum, mea quidem ſententia, partem conſtituere debeat Curſus matheſeos elementaris, eo, quo oportet, rigore & ambitu concinnati; ſuperſediſſem huic operæ, ſi quem noſſem, ad quem remittere potuiſſem. Ceterum comparatio attenta docebit, nullam in Capite hoc, utut amplo, tradi propoſitionem, quin ad applicationes ſequentes neceſſaria ſit.

Deſiderio ſolo utile quid præſtandi motus non reformido reprehenſionem; quod ſententia, quam ſuſtineo, nova non ſit, ſed ab variis jam mathematicis propoſita. Gnarus ſcriptorum D'ALEMBERTII, COUSINI, KÆSTNERI, KARSTENII, TEMPELHOFFII, PASQUICHII, præſertim diſſertationum egregiarum ROBINSII, & tractatus ſolidi MACLAURINI, in animum haud induxiſſem meditationes meas de arduo hoc objecto publice exponere; niſi quæſtio ab Societate literaria adeo illuſtri promulgata attentionem meam excitaſſet, mihique perſuaſiſſet, quæ jam præſtita ſint, nondum omnibus deſideriis ſatisfacere ipſi videri.

Dubium non eſt, quin NEWTONUS doctrinas ſuas iisdem principiis ſuperſtruere, ſpeciatim per rationes primas & ultimas, tam frequenter in Principiis ſuis uſurpatas, limites rationum intelligi voluerit; cum ipſe in Scholio Lemmati XI. Libri I. Princip. ſubjuncto monuerit: *Ultimæ rationes illæ, quibuscum „quantitates evaneſcunt, revera non ſunt rationes quantitatum ultimarum; ſed limites, ad „quos quantitatum ſine limite decreſcentium rationes ſemper appropinquant, & quos pro„pius aſſequi poſſunt quam pro data quavis differentia.*

Quam-

Quamvis Leibnitius modis loquendi magis audacibus uti aſſueverit: eos tamen explicatione rigidandos eſſe præcepit (Act. Erud. Lipſ. 1712. p. 167.); & momentum demonſtrationum exactarum agnovit atque commendavit. Ad Joh. Bernoullium ſcribens (31 Dec. 1700.) de Nieuwentitii, Rollii, Cluverii, & aliorum adverſus calculum differentialem objectionibus: *Perutile* „*eſt*, inquit, *os illis occludi per reductionem ad demonſtrationes veterum more formatas*. (Leibnitii & Bernoullii Commerc. philos. & math. T. II. Ep. 108.)

Et quamvis Wolfius, veſtigiis Leibnitii inſiſtens, infinitum profuſe in ſcriptis ſuis ſparſerit; methodo tamen Archimedeæ, quam jure ſibi vindicat, prærogativam tribuit: „Ipſius (Archimedis) demonſtrandi methodo principia „methodi infiniteſimalis rigidantur." (Elem. Matheſ. univ. T. I. Geom. Theor. 86. Schol.)

Pluribus aliis auctoritatibus poſſem inſtitutum meum munire; atque ex. gr. conſenſum ejus cum methodo indiviſibilium, ſano ſenſu explicationibusque ſagaciſſimi ipſius auctoris conformiter intellecta, docere: verum hæc longius me ab ſcopo propoſito abducerent. Quare, illis miſſis, ad partem Introductionis hujus mathematicam, ſequentibus quatuor Capitibus traditam, progredior.

Tubingæ, 1 Mart. 1795.

Caput

Caput A.

Theorematis binomialis Newtoni *demonſtratio generalis elementaris.*

Theorema binomiale Newtoni univerſim ad quoscunque exponentes, poſitivos & negativos, integros & fractos, extenſum adeo uberis eſt per totam matheſin uſus, ut demonſtrationem ejus firmam & elementarem ſummi momenti eſſe cenſeam; eoque magis, quod Vir celeberrimus, a quo denominari conſuevit, & plurimi inſignes mathematici poſt ipſum, ſola, ut videtur, analogia fulti, formulam, de exponentibus integris ac poſitivis tantum eo, quo par eſt, rigore demonſtratam, ad exponentes fractos & negativos applicare acquieverunt. In Diſſertatione mea, inſcripta: *Expoſition elementaire des principes des calculs ſuperieurs*, p. 26. ejusmodi demonſtrationis compotem me eſſe aſſerui; ſed brevitatis ſtudio eam omittere conſultum duxi. Quæ cum fuerit, ut non omnino facilis, in Diario litterario Gœttingenſi (Nro. 165. 16 Oct. 1788.) deſiderata; in hac introductione illam exponere non dubitavi. Conſentit ea præter leviores quasdam mutationes & illuſtrationes, quas neceſſarias eſſe cenſui, cum methodo, quam in Commentariis Acad. Reg. Berol. ad annum 1777. tradidit illuſtris Segnerus; &, quo debet, modo evoluta mea quidem ſententia tam propoſito ſatisfacit, quam ea ſe brevitate commendat, cujus demonſtratio mere elementaris capax eſſe poteſt.

§. *a.* *Theorema.* Sint duæ quantitates P & P', quales ſequuntur

$$P=a^n+\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+\frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}b^4+\frac{n}{1}\cdots\frac{n-4}{5}a^{n-5}b^5+\ldots$$

$$P'=a^{n'}+\frac{n'}{1}a^{n'-1}b+\frac{n'}{1}\cdot\frac{n'-1}{2}a^{n'-2}b^2+\frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3}a^{n'-3}b^3+\frac{n'}{1}\cdots\frac{n'-3}{4}a^{n'-4}b^4+\frac{n'}{1}\cdots\frac{n'-4}{5}a^{n'-5}b^5+\ldots$$

Dico: productum ex hisce duabus quantitatibus conflatum ejusdem eſſe formæ, cujus utraque illarum, ſi loco τȣ n aut n' illarum ſumma $n+n'$ ſubſtituatur. Nempe

$$PP'=a^{n+n'}+\frac{n+n'}{1}a^{n+n'-1}b+\frac{n+n'}{1}\cdot\frac{n+n'-1}{2}a^{n+n'-2}b^2+\frac{n+n'}{1}\cdots\frac{n+n'-2}{3}a^{n+n'-3}b^3+$$
$$\frac{n+n'}{1}\cdots\frac{n+n'-3}{4}a^{n+n'-4}b^4+\frac{n+n'}{1}\cdots\frac{n+n'-4}{5}a^{n+n'-5}b^5+\ldots$$

Et primo quidem exponentes τȣ *a* in ſucceſſivis producti terminis progreſſionem arithmeticam decreſcentem ſequuntur, cujus primus terminus eſt $n+n'$, & communis terminorum differentia eſt unitas. Exponentes autem τȣ *b* ſequuntur progreſſionem numerorum naturalium creſcentem inde ab unitate.

Determinanda ſupereſt lex coëfficientium.

1°. Coëfficiens primi termini eſt unitas.

2°. Coëfficiens ſecundi termini eſt $\frac{n}{1}+\frac{n'}{1}=\frac{n+n'}{1}$.

3°. Coëfficiens termini tertii eſt

$$\left.\begin{matrix} \frac{n}{1}\cdot\frac{n-1}{2} \\ +\frac{n}{1}\cdot\frac{n'}{1} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\end{matrix}\right\} = \left\{\begin{matrix} \frac{n}{1}\cdot\frac{n-1}{2} \\ +2\frac{n}{1}\cdot\frac{n'}{2} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\end{matrix}\right. = \left\{\begin{matrix} \frac{n}{1}\cdot\frac{n-1}{2}+\frac{n}{1}\cdot\frac{n'}{2} \\ +\frac{n}{1}\cdot\frac{n'}{2}+\frac{n'}{1}\cdot\frac{n'-1}{2}\end{matrix}\right. = \left\{\begin{matrix}\frac{n+n'-1}{2}\cdot\frac{n}{1} \\ +\frac{n+n'-1}{2}\cdot\frac{n'}{1}\end{matrix}\right. = \frac{n+n'}{1}\cdot\frac{n+n'-1}{2} \quad (2°.)$$

4°. Coëfficiens termini quarti eſt

$$\left.\begin{matrix} \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} \\ +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1} \\ +\frac{n}{1}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3}\end{matrix}\right\} = \left\{\begin{matrix} \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} \\ +3\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3} \\ +3\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\frac{n'-1}{3} \\ +\frac{n'}{1}\cdot\frac{n'-1}{2}\cdot\frac{n'-2}{3}\end{matrix}\right. = \frac{n+n'-2}{3}\left(\begin{matrix}\frac{n}{1}\cdot\frac{n-1}{2} \\ +2\cdot\frac{n}{1}\cdot\frac{n'}{2} \\ +1\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\end{matrix}\right) = \frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\frac{n+n'-2}{3} \quad (3°.)$$

5°. Coëfficiens termini quinti eſt

$$\left.\begin{matrix} \frac{n}{1}\cdots\cdots\frac{n-3}{4} \\ +\frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{n'}{1} \\ +\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\ +\frac{n}{1}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3} \\ +\frac{n'}{1}\cdots\cdots\frac{n'-3}{4}\end{matrix}\right\} = \left\{\begin{matrix} \frac{n}{1}\cdots\cdots\frac{n-3}{4} \\ +4\cdot\frac{n}{1}\cdots\frac{n-3}{3}\cdot\frac{n'}{4} \\ +6\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2} \\ +4\cdot\frac{n}{1}\cdot\frac{n'}{1}\cdots\frac{n'-2}{3} \\ +1\cdot\frac{n'}{1}\cdots\cdots\frac{n'-3}{4}\end{matrix}\right. = \frac{n+n'-3}{4}\left(\begin{matrix}\frac{n}{1}\cdots\cdots\frac{n-2}{3} \\ +3\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3} \\ +3\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\frac{n'-1}{3} \\ +\frac{n'}{1}\cdots\frac{n'-2}{3}\end{matrix}\right) = \frac{n+n'}{1}\cdots\frac{n+n'-3}{4} \quad (4°.)$$

6°. Coëfficiens termini ſexti eſt

$$
\left.\begin{array}{l}
\frac{n}{1}\cdot\ldots\cdot\frac{n-4}{5}\\
+\frac{n}{1}\ldots\frac{n-3}{4}\cdot\frac{n'}{1}\\
+\frac{n}{1}\ldots\frac{n-2}{3}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\ldots\frac{n'-2}{3}\\
+\frac{n}{1}\cdot\frac{n'}{1}\ldots\frac{n'-3}{4}\\
+\frac{n'}{1}\cdot\ldots\cdot\frac{n'-4}{5}
\end{array}\right\}
=\left\{\begin{array}{l}
\frac{n}{1}\cdot\ldots\cdot\frac{n-4}{5}\\
+5\cdot\frac{n}{1}\ldots\frac{n-3}{4}\cdot\frac{n'}{5}\\
+10\cdot\frac{n}{1}\ldots\frac{n-2}{3}\cdot\frac{n'}{4}\cdot\frac{n'-1}{5}\\
+10\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\ldots\frac{n'-2}{5}\\
+5\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\ldots\cdot\frac{n'-3}{5}\\
+1\cdot\frac{n'}{1}\cdot\ldots\cdot\frac{n'-4}{5}
\end{array}\right.
=\frac{n+n'-4}{5}\left[\begin{array}{l}
\frac{n}{1}\ldots\frac{n-3}{4}\\
+4\cdot\frac{n}{1}\ldots\frac{n-2}{3}\cdot\frac{n'}{4}\\
+6\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdot\frac{n'-1}{4}\\
+4\cdot\frac{n}{1}\cdot\frac{n'}{2}\ldots\frac{n'-2}{4}\\
+\frac{n'}{1}\cdot\ldots\cdot\frac{n'-3}{4}
\end{array}\right]
=\frac{n+n'}{1}\ldots\frac{n+n'-4}{5}\quad(5^\circ.)
$$

Generatim, oſtenſo, quod lex valeat pro termino quocunque m^{to}, ſcilicet quod coëfficiens m^{ti} termini eſt $\frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\ldots\cdot\frac{n+n'-(m-2)}{m-1}$: dico, eandem legem etiam valere pro termino ſequenti $m+1^{to}$; ſcilicet coëfficientem hujus termini eſſe $\frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\ldots\cdot\frac{n+n'-(m-1)}{m}$.

Etenim coëfficiens $m+1^{ti}$ termini eſt

$$
\left.\begin{array}{l}
\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\cdot\frac{n-(m-1)}{m}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-2)}{m-1}\cdot\frac{n'}{1}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{1}\ldots\frac{n'-2}{3}\\
+\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{1}\ldots\frac{n'-3}{4}\\
\quad\vdots\\
+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\ldots\frac{n'-(m-3)}{m-2}\\
+\frac{n}{1}\cdot\frac{n'}{1}\cdot\ldots\cdot\frac{n'-(m-2)}{m-1}\\
+\frac{n'}{1}\cdot\ldots\cdot\frac{n'-(m-1)}{m}
\end{array}\right\}
=\left\{\begin{array}{l}
\frac{n}{1}\cdot\ldots\cdot\frac{n-(m-1)}{m}\\
+\frac{m}{1}\cdot\frac{n}{1}\cdot\ldots\cdot\frac{n-(m-2)}{m-1}\cdot\frac{n'}{m}\\
+\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{n}{1}\ldots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{m-1}\cdot\frac{n'-1}{m}\\
+\frac{m}{1}\ldots\frac{m-2}{3}\cdot\frac{n}{1}\ldots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{m-2}\ldots\frac{n'-2}{m}\\
+\frac{m}{1}\ldots\frac{m-3}{4}\cdot\frac{n}{1}\ldots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{m-3}\ldots\frac{n'-3}{m}\\
\quad\vdots\\
+\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\cdot\ldots\cdot\frac{n'-(m-3)}{m}\\
+\frac{m}{1}\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\ldots\cdot\frac{n'-(m-2)}{m}\\
+\frac{n'}{1}\cdot\ldots\cdot\frac{n'-(m-1)}{m}
\end{array}\right.
$$

$= n+n$

$$= \frac{n+n'-(m-1)}{m}\left\{\begin{aligned}
&\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-2)}{m-1}\\
&+\frac{m-1}{1}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{m-1}\\
&+\frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{m-2}\cdot\frac{n'-1}{m-1}\\
&+\frac{m-1}{1}\ldots\frac{m-3}{3}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\ldots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{m-3}\ldots\frac{n'-2}{m-1}\\
&\quad\vdots\\
&+\frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{3}\ldots\frac{n'-(m-4)}{m-1}\\
&+\frac{m-1}{1}\cdot\frac{n}{1}\cdot\frac{n'}{2}\cdot\ldots\ldots\frac{n'-(m-3)}{m-1}\\
&+\ 1\cdot\frac{n'}{1}\cdot\ldots\ldots\frac{n'-(m-2)}{m-1}
\end{aligned}\right)$$

$$= \frac{n+n'-(m-1)}{m}\left\{\begin{aligned}
&\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-2)}{m-1}\\
&+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-3)}{m-2}\cdot\frac{n'}{1}\\
&+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-4)}{m-3}\cdot\frac{n'}{1}\cdot\frac{n'-1}{2}\\
&+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\ldots\ldots\frac{n-(m-5)}{m-4}\cdot\frac{n'}{1}\ldots\frac{n'-2}{3}\\
&\quad\vdots\\
&+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n'}{1}\cdot\ldots\ldots\frac{n'-(m-4)}{m-3}\\
&+\frac{n}{1}\cdot\frac{n'}{1}\cdot\ldots\ldots\frac{n'-(m-3)}{m-2}\\
&+\frac{n'}{1}\cdot\ldots\ldots\frac{n'-(m-2)}{m-1}
\end{aligned}\right)$$

$$= \frac{n+n'}{1}\cdot\frac{n+n'-1}{2}\cdot\frac{n+n'-2}{3}\cdot\ldots\ldots\frac{n+n'-(m-1)}{m}\ \text{(per hyp.).}$$

Ergo ſi lex coëfficientium valet pro quocunque producti termino, eadem quoque valet pro coëfficiente termini ſequentis. Atqui valere demonſtrata fuit in termino ſecundo, tertio, quarto, quinto, ſexto: valet igitur etiam in ſeptimo, atque hinc rurſus in octavo, tum in nono, & ſic conſequenter in univerſa ſerie.

§. *b.* *Applicatio prima.* 1°. Quadratum quantitatis hujus formæ

$$a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots$$

eſt $a^{2n}+\frac{2n}{1}a^{2n-1}b+\frac{2n}{1}\cdot\frac{2n-1}{2}a^{2n-2}b^2+\frac{2n}{1}\ldots\frac{2n-2}{3}a^{2n-3}b^3+\frac{2n}{1}\ldots\frac{2n-3}{4}a^{2n-4}b^4+\ldots$

2°. Hinc cubus ejusdem quantitatis eſt

$$a^{3n}+\frac{3n}{1}a^{3n-1}b+\frac{3n}{1}\cdot\frac{3n-1}{2}a^{3n-2}b^2+\frac{3n}{1}\ldots\frac{3n-2}{3}a^{3n-3}b^3+\frac{3n}{1}\ldots\frac{3n-3}{4}a^{3n-4}b^4+\ldots$$

Hinc iterum 3°. potentia quarta prædictæ quantitatis eſt

$$a^{4n}+\frac{4n}{1}a^{4n-1}b+\frac{4n}{1}\cdot\frac{4n-1}{2}a^{4n-2}b^2+\frac{4n}{1}\ldots\frac{4n-2}{3}a^{4n-3}b^3+\frac{4n}{1}\ldots\frac{4n-3}{4}a^{4n-4}b^4+\ldots$$

Et univerſim, quicunque numerus integer poſitivus fuerit m, potentia m^{ta} ejusdem quantitatis eſt

$$a^{mn}+\frac{mn}{1}a^{mn-1}b+\frac{mn}{1}\cdot\frac{mn-1}{2}a^{mn-2}b^2+\frac{mn}{1}\ldots\frac{mn-2}{3}a^{mn-3}b^3+\frac{mn}{1}\ldots\frac{mn-3}{4}a^{mn-4}b^4+\ldots$$

§. *c.* *Applicatio ſecunda.* Viciſſim, denotante m numerum quemcunque integrum poſitivum, radix m^{ta} quantitatis

$$a^n+\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2+\frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3+\frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4+\ldots$$

eſt $a^{\frac{n}{m}}+\frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4+\ldots$

Etenim (per applicationem præcedentem) prior quantitas eſt potentia m^{ta} poſterioris: quare viciſſim poſterior quantitas eſt radix m^{ta} prioris.

§. *d.* *Applicatio tertia.* Sint m & n numeri quilibet integri poſitivi. Dico eſſe

$$(a+b)^{\frac{n}{m}}=a^{\frac{n}{m}}+\frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b+\frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3+\frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4+\ldots$$

Etenim $(a+b)^{\frac{n}{m}} = \sqrt[m]{(a+b)^n}$

$$= \sqrt[m]{\left(a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots\right)} \quad (\S.\ b.)$$

$$= a^{\frac{n}{m}} + \frac{\frac{n}{m}}{1}a^{\frac{n}{m}-1}b + \frac{\frac{n}{m}}{1}\cdot\frac{\frac{n}{m}-1}{2}a^{\frac{n}{m}-2}b^2 + \frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-2}{3}a^{\frac{n}{m}-3}b^3 + \frac{\frac{n}{m}}{1}\ldots\frac{\frac{n}{m}-3}{4}a^{\frac{n}{m}-4}b^4 + \ldots \quad (\S.\ c.)$$

§. *e.* *Corollarium.* Univerſim igitur, ſi n fuerit numerus rationalis poſitivus quicunque, ſive integer, ſive fractus,

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots$$

§. *f.* *Applicatio quarta.* Sint duæ quantitates

$$a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots$$

$$a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\ldots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\ldots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

Productum ex illis factum reducitur ad primum terminum $a^n \times a^{-n} = 1$: cum terminorum omnium ſequentium coëfficientes ingrediatur factor $n-n = 0$.

§. *g.* *Corollarium.* Ergo

$$a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\ldots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\ldots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

$$= \frac{1}{a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}a^{n-4}b^4 + \ldots}$$

Atqui, n denotante numerum quemcunque rationalem poſitivum, denominator hujus fractionis eſt $(a+b)^n$ (§. *e*). Ergo

$$\frac{1}{(a+b)^n} \text{ ſeu } (a+b)^{-n} = a^{-n} + \frac{-n}{1}a^{-n-1}b + \frac{-n}{1}\cdot\frac{-n-1}{2}a^{-n-2}b^2 + \frac{-n}{1}\ldots\frac{-n-2}{3}a^{-n-3}b^3 + \frac{-n}{1}\ldots\frac{-n-3}{4}a^{-n-4}b^4 + \ldots$$

$$= \frac{1}{a^n} - \frac{n}{1}\frac{b}{a^{n+1}} + \frac{n}{1}\cdot\frac{n+1}{2}\frac{b^2}{a^{n+2}} - \frac{n}{1}\ldots\frac{n+2}{3}\frac{b^3}{a^{n+3}} + \frac{n}{1}\ldots\frac{n+3}{4}\frac{b^4}{a^{n+4}} - \ldots$$

Eadem itaque lex, quæ valet de exponentibus rationalibus poſitivis, pariter applicatur ad exponentes rationales negativos.

§. *h.* *Obſervatio.* Quod ad exponentes irrationales attinet: cum per extractionem radicum, vero propius ſemper propiusque accedentem, quantitates ſurdæ exprimi poſſint per quantitates rationales, quæ a vero illarum valore minus quam quantitate data differant; ad exponentes ſurdos applicare licet, quæ de exponentibus rationalibus demonſtrata fuerunt.

Obſervandum præterea: calculos, quos quantitates exponentibus ſurdis affectæ ingrediuntur, nonniſi per analogiam juxta regulas exponentium rationalium peragi.

Quod ipſum tanto etiam magis ad exponentes imaginarios pertinet; cum mera ſint ſigna, facilitatis & univerſalitatis calculi cauſa ab mathematicis introducta.

Caput B.

De differentiis quantitatum mutabilium.

§. *i.* *Definitio.* Sit aliqua quantitas mutabilis, quomodocunque expreſſa per alteram quantitatem mutabilem, ſolam, vel cum quantitatibus conſtantibus combinatam. Prior quantitas dicitur *functio* poſterioris.

Quod attinet ad functionum diviſionem in integras & fractas, uniformes & multiformes, rationales & irrationales, algebraicas & transcendentes; videatur inter alios Euleri *Introductio*, Cap. I.

§. *k.* Sit P functio quantitatis variabilis x uniformiter creſcentis vel decreſcentis. Denotentur incrementa vel decrementa ejus ſucceſſiva ſignis

$\Delta x,\ 2\Delta x,\ 3\Delta x,\ 4\Delta x,\ \ldots\ (n-1)\Delta x,\ n\Delta x.$

Tunc, qui valoribus ipſius ſucceſſivis	$x+\Delta x,\ x+2\Delta x,\ x+3\Delta x,\ x+4\Delta x \ldots x+(n-1)\Delta x,\ x+n\Delta x$
reſpondent functionis P valores, deſignentur per	$P',\quad P'',\quad P''',\quad P^{iv} \ldots P^{N-1},\quad P^{N}.$
Et valorum horum mutationes ſucceſſivæ ita notentur	$\Delta P,\quad \Delta P',\quad \Delta P'',\quad \Delta P''' \ldots \Delta P^{N-11},\quad \Delta P^{N-1}.$

Erunt ideo

$$\begin{aligned} P' &= P + \Delta P \\ P'' &= P' + \Delta P' \\ P''' &= P'' + \Delta P'' \\ P^{iv} &= P''' + \Delta P''' \\ &\vdots \\ P^{N-1} &= P^{N-11} + \Delta P^{N-11} \\ P^{N} &= P^{N-1} + \Delta P^{N-1}. \end{aligned}$$

Quan-

Quantitates ΔP, $\Delta P'$, $\Delta P''$, $\Delta P'''$ . . . , ΔP^{N-II}, ΔP^{N-I} vocantur *differentiæ primi ordinis* quantitatis P.

§. *l.* Si hæ differentiæ non fuerint conſtantes: eædem componentur ex quantitate conſtante Δx, & ex variabili x; adeoque pariter ſunt functiones variabilis x. Succeſſivæ harum functionum mutationes, uniformi quantitatis x mutationi reſpondentes, deſignentur per $\Delta^2 P$, $\Delta^2 P'$, $\Delta^2 P''$, $\Delta^2 P'''$ . . . $\Delta^2 P^{N-II}$, $\Delta^2 P^{N-I}$.

Hæ differentiæ differentiarum primi ordinis vocantur *differentiæ ſecundi ordinis* functionis P; & eodem quo prius modo obtinentur æquationes ſequentes:

$$\begin{aligned}
\Delta P' &= \Delta P + \Delta^2 P \\
\Delta P'' &= \Delta P' + \Delta^2 P' \\
\Delta P''' &= \Delta P'' + \Delta^2 P'' \\
&\vdots \\
\Delta P^{N-I} &= \Delta P^{N-II} + \Delta^2 P^{N-II} \\
\Delta P^{N} &= \Delta P^{N-I} + \Delta^2 P^{N-I}.
\end{aligned}$$

§. *m.* Si differentiæ ſecundi ordinis non fuerint conſtantes: pariter componentur ex quantitate conſtanti Δx^2 & variabili x; ideoque functiones erunt quantitatis x. Succeſſivæ earum mutationes, uniformi ipſius x mutationi reſpondentes, deſignentur per $\Delta^3 P$, $\Delta^3 P'$, $\Delta^3 P''$, $\Delta^3 P'''$ . . . $\Delta^3 P^{N-II}$, $\Delta^3 P^{N-I}$. Differentiæ iſtæ differentiarum ſecundi ordinis vocantur *differentiæ tertii ordinis*; & rurſus ſunt

$$\begin{aligned}
\Delta^2 P' &= \Delta^2 P + \Delta^3 P \\
\Delta^2 P'' &= \Delta^2 P' + \Delta^3 P' \\
\Delta^2 P''' &= \Delta^2 P'' + \Delta^3 P'' \\
\Delta^2 P^{IV} &= \Delta^2 P''' + \Delta^3 P''' \\
&\vdots \\
\Delta^2 P^{N-I} &= \Delta^2 P^{N-II} + \Delta^3 P^{N-II} \\
\Delta^2 P^{N} &= \Delta^2 P^{N-I} + \Delta^3 P^{N-I}.
\end{aligned}$$

§. *n.* Si differentiæ tertii ordinis non ſunt conſtantes; progreſſus fit ad earundem differentias, ſeu ad *differentias quarti ordinis:* deinde ſimiliter ad *differentias quinti ordinis*, & hinc ad *differentias ſexti ordinis*, atque ita deinceps; donec (ſi fieri poſſit) perveniatur ad *differentias conſtantes*, ſeu quarum nullæ ſunt differentiæ.

Gene-

Generatim differentiæ differentiarum $m-1^{ti}$ ordinis vocantur *differentiæ* m^{ti} *ordinis;* & ſic deſignantur $\Delta^m P$, $\Delta^m P'$, $\Delta^m P''$, $\Delta^m P'''$ . . . $\Delta^m P^{N-II}$, $\Delta^m P^{N-I}$. Unde

$$\begin{aligned}
\Delta^{m-1}P' &= \Delta^{m-1}P + \Delta^m P \\
\Delta^{m-1}P'' &= \Delta^{m-1}P' + \Delta^m P' \\
\Delta^{m-1}P''' &= \Delta^{m-1}P'' + \Delta^m P'' \\
\Delta^{m-1}P^{IV} &= \Delta^{m-1}P''' + \Delta^m P''' \\
&\vdots \\
\Delta^{m-1}P^{N-I} &= \Delta^{m-1}P^{N-II} + \Delta^m P_{N-II} \\
\Delta^{m-1}P^{N} &= \Delta^{m-1}P^{N-I} + \Delta^m P^{N-I}.
\end{aligned}$$

§. 0. Differentiæ omnium ordinum ſucceſſivorum functionis P immediate per ſeriem functionum P, P', P'', P''' . . . P^{N-I}, P^N exprimi poſſunt modo ſequenti:

$$\begin{aligned}
\Delta P &= P' - P \\
\Delta P' &= P'' - P' \\
\text{Ergo } \Delta P' - \Delta P \text{ ſeu } \Delta^2 P &= P'' - 2P' + P \\
\Delta^2 P' &= P''' - 2P'' + P' \\
\text{Ergo } \Delta^3 P &= P''' - 3P'' + 3P' - P \\
\Delta^3 P' &= P^{IV} - 3P''' + 3P'' - P' \\
\text{Ergo } \Delta^4 P &= P^{IV} - 4P''' + 6P'' - 4P' + P \\
\Delta^4 P' &= P^{V} - 4P^{IV} + 6P''' - 4P'' + P' \\
\text{Ergo } \Delta^5 P &= P^{V} - 5P^{IV} + 10P''' - 10P'' + 5P' - P.
\end{aligned}$$

Generaliter

ſit $\Delta^m P = P^M - \frac{m}{1} P^{M-I} + \frac{m}{1} \cdot \frac{m-1}{2} P^{M-II} - \frac{m}{1} \dots \frac{m-2}{3} P^{M-III} + \dots \mp mP' \pm P$

& ideo

$\Delta^m P' = P^{M+I} - \frac{m}{1} P^M + \frac{m}{1} \cdot \frac{m-1}{2} P^{M-I} - \frac{m}{1} \dots \frac{m-2}{3} P^{M-II} + \frac{m}{1} \dots \frac{m-3}{4} P^{M-III} - \dots \pm P'$

erit

$\Delta^{m+1} P = P^{M+I} - \frac{m+1}{1} P^M + \frac{m+1}{1} \cdot \frac{m}{2} P^{M-I} - \frac{m+1}{1} \dots \frac{m-1}{3} P^{M-II} + \frac{m+1}{1} \dots \frac{m-2}{4} P^{M-III} - \dots \pm \frac{m+1}{1} P' \mp P.$

 Ergo

Ergo generaliter

$$\Delta^m P = P^M - \frac{m}{1} P^{M-1} + \frac{m}{1} \cdot \frac{m-1}{2} P^{M-II} - \frac{m}{1} \ldots \frac{m-2}{3} P^{M-III} \ldots \pm \frac{m}{1} \cdot \frac{m-1}{2} P'' \mp \frac{m}{1} P' \pm P.$$

§. *p*. Viciſſim terminus quilibet P^M ſeriei P, P', P'', P''' ... P^{M-1}, P^M exprimi poteſt per functionem P, & per differentias ſucceſſivas omnium ordinum hujus functionis P.

Etenim quoniam

$$\begin{aligned}
&P' &&= P+\Delta P\\
&\Delta P' &&= \Delta P+\Delta^2 P\\
\text{ergo}\quad &P''(=P'+\Delta P') &&= P+2\Delta P+\Delta^2 P\\
\text{Hinc}\quad &\Delta P'' &&= \Delta P+2\Delta^2 P+\Delta^3 P\\
\text{ergo}\quad &P'''(=P''+\Delta P'') &&= P+3\Delta P+3\Delta^2 P+\Delta^3 P\\
\text{Hinc}\quad &\Delta P''' &&= \Delta P+3\Delta^2 P+3\Delta^3 P+\Delta^4 P\\
\text{ergo}\quad &P^{IV}(=P'''+\Delta P''') &&= P+4\Delta P+6\Delta^2 P+4\Delta^3 P+\Delta^4 P\\
\text{Hinc}\quad &\Delta P^{IV} &&= \Delta P+4\Delta^2 P+6\Delta^3 P+4\Delta^4 P+\Delta^5 P\\
\text{ergo}\quad &P^{V}(=P^{IV}+\Delta P^{IV}) &&= P+5\Delta P+10\Delta^2 P+10\Delta^3 P+5\Delta^4 P+\Delta^5 P.
\end{aligned}$$

Generatim ſit

$$P^M = P + \frac{m}{1}\Delta P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^2 P + \frac{m}{1}\ldots\frac{m-2}{3}\Delta^3 P + \frac{m}{1}\ldots\frac{m-3}{4}\Delta^4 P + \ldots + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^{m-2}P + \frac{m}{1}\Delta^{m-1}P + \Delta^m P$$

hinc

$$\Delta P^M = \Delta P + \frac{m}{1}\Delta^2 P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^3 P + \frac{m}{1}\ldots\frac{m-2}{3}\Delta^4 P + \ldots + \frac{m}{1}\ldots\frac{m-2}{3}\Delta^{m-2}P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^{m-1}P + \frac{m}{1}\Delta^m P + \Delta^{m+1}P.$$

erit

$$P^{M+1}(=P^M+\Delta P^M) = P + \frac{m+1}{1}\Delta P + \frac{m+1}{1}\cdot\frac{m}{2}\Delta^2 P + \frac{m+1}{1}\ldots\frac{m-1}{3}\Delta^3 P + \frac{m+1}{1}\ldots\frac{m-2}{4}\Delta^4 P + \ldots + \frac{m+1}{1}\ldots\frac{m-1}{3}\Delta^{m-2}P + \frac{m+1}{1}\cdot\frac{m}{2}\Delta^{m-1}P + \frac{m+1}{1}\Delta^m P + \Delta^{m+1}P.$$

Ergo generatim

$$P^M = P + \frac{m}{1}\Delta P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^2 P + \frac{m}{1}\ldots\frac{m-2}{3}\Delta^3 P + \frac{m}{1}\ldots\frac{m-3}{4}\Delta^4 P + \ldots + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^{m-2}P + \frac{m}{1}\Delta^{m-1}P + \Delta^m P.$$

Propoſitio hæc magni eſt momenti ad applicationes, quæ deinceps tradentur.

§. *q*.

§. 9. Cum functio quælibet quantitatis variabilis potentiis variabilis hujus (immediate aut mediate) exprimi poſſit: multum intereſt, potentias quantitatis variabilis & differentias omnium ordinum harum potentiarum accuratius expendere. Ex ſequentibus patebit, quanti ſpeciatim hoc reſpectu momenti ſit contemplatio potentiarum exponentis integri poſitivi cujuscunque.

Theorema. Potentiæ quantitatis variabilis, cujus exponens eſt numerus integer poſitivus, differentia ordinis, cujus index idem eſt cum exponente poteſtatis, æquatur producto continuo numerorum naturalium inde ab unitate usque ad hunc exponentem, ducto in differentiæ quantitatis variabilis potentiam ejusdem exponentis: proindeque differentiæ ordinum ſuperiorum ejusdem potentiæ quantitatis variabilis evaneſcunt.

Sit $P = x^m$, exponente m exiſtente numero integro poſitivo.

Dico eſſe $\Delta^m P = 1.2.3.4 \ldots m \Delta x^m$; & proinde $\Delta^{m+r} P = 0$.

Ad demonſtrandum hoc theorema oſtendam primo: illud verum eſſe pro minoribus exponentis m valoribus, quales ſunt 1, 2, 3, 4 .... Tum evincam: quod, ſi theorema verum fuerit pro quolibet exponente dato, idem etiam valeat pro exponente ſequenti, qui priorem unitate ſuperet.

Exemplum 1. Sit $P = x$

$$\Delta P = \Delta x = 1.\Delta x$$

Corollarium. $\Delta^{r+1} x = 0.$

Exemplum 2. Sit $P = x^2$

$$\Delta P = 2x\Delta x + \Delta x^2$$

$$\Delta^2 P = 2\Delta x \times \Delta x + \Delta x^2 \times \Delta 1$$

$$= 2\Delta x^2 = 1.2.\Delta x^2.$$

Coroll. $\Delta^{r+2} x = 0.$

Exemplum 3. Sit $P = x^3$

$$\Delta P = 3xx\Delta x + 3x\Delta x^2 + \Delta x^3$$

$$\Delta^3 P (= \Delta^2 \Delta P) = 3\Delta x.\Delta^2 xx + 3\Delta x^2.\Delta^2 x + \Delta x^3.\Delta^2 1$$

$$= 3\Delta x.\Delta^2 xx = 1.2.3\Delta x^3$$

Coroll. $\Delta^{r+3} x^3 = 0.$

Exem-

Exemplum 4. Sit

$$P = x^4$$

$$\Delta P = 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4$$

$$\Delta^4 P (= \Delta^3 . \Delta P) = 4\Delta x \times \Delta^3 x^3 + 6\Delta x^2 . \Delta^3 x^2 + 4\Delta x^3 . \Delta^3 x + 4\Delta x^4 . \Delta^3 . 1$$

$$= 4\Delta x \times \Delta^3 x^3 = 1.2.3.4\Delta x^4.$$

Coroll. $\Delta^{1+4} x^4 = 0.$

Generatim ſit m numerus integer poſitivus, & $P = x^m$. Probatum fuerit, quod $\Delta^m P = 1.2.3.4 \ldots m\Delta x^m$ & $\Delta^m x^{m-1} = 0$. Dico, quod $\Delta^{m+1} x^{m+1} = 1.2.3 \ldots (m+1)\Delta x^{m+1}$ & $\Delta^{m+1+1} x^{m+1} = 0$.

Etenim

$$\Delta x^{m+1} = \frac{m+1}{1} x^m \Delta x + \frac{m+1}{1}.\frac{m}{2} x^{m-1}\Delta x^2 + \frac{m+1}{1}\ldots\frac{m-1}{3} x^{m-2}\Delta x^3 + \frac{m+1}{1}\ldots\frac{m-2}{4} x^{m-3}\Delta x^4 + \ldots + \frac{m+1}{1}.\frac{m}{2} x^2 \Delta x^{m-1} + \frac{m+1}{1} x\Delta x^m + \Delta x^{m+1}$$

Ergo

$$\Delta^{m+1} x^{m+1} = \overline{m+1} . \Delta x \times \Delta^m x^m + \frac{m+1}{1}.\frac{m}{2}\Delta x^2 \times \Delta^m x^{m-1} + \frac{m+1}{1}\ldots\frac{m-1}{3}\Delta x^3 \times \Delta^m x^{m-2} + \frac{m+1}{1}\ldots\frac{m-2}{4}\Delta x^4 \times \Delta^m x^{m-3} + \ldots + \frac{m+1}{1}.\frac{m}{2}\Delta x^{m-1} \times \Delta^m x^2 + \frac{m+1}{1}\Delta x^m \times \Delta^m x + \Delta x^{m+1} \times \Delta^m 1$$

$$= (m+1)\Delta x \times \Delta^m x^m \quad \text{(hyp.)}$$

$$= (m+1)\Delta x \times 1.2.3.4 \ldots \overline{m-1} . m\Delta x^m \quad \text{(hyp.)}$$

$$= 1.2.3.4 \ldots (m+1)\Delta x^{m+1}$$

Corollarium. $\Delta^{m+1+1} x^{m+1} = 0.$

Ergo generaliter

$$\Delta^m x^m = 1.2.3.4 \ldots m\Delta x^m$$

$$\& \quad \Delta^{m+1} x^m = 0.$$

§. r. *Corollarium.* Speciatim poteſtatum numerorum naturalium ſucceſſivorum, quarum exponentes ſunt integri poſitivi, differentiæ, quarum indices ordinum ſunt exponentibus his æquales, æquantur producto continuo numerorum naturalium ab unitate inde usque ad illum exponentem; earundemque potentiarum differentiæ, quarum index ordinis exponentem illum ſuperat, evaneſcunt.

Scili-

Scilicet $\Delta^1 n^1 = 1$ $\Delta^{r+1} n^1 = 0$

$\Delta^2 n^2 = 1.2$ $\Delta^{r+2} n^2 = 0$

$\Delta^3 n^3 = 1.2.3$ $\Delta^{r+3} n^3 = 0$

$\Delta^4 n^4 = 1.2.3.4$ $\Delta^{r+4} n^4 = 0$

$\vdots$ $\vdots$

$\Delta^m n^m = 1.2.3 \ldots . m$ $\Delta^{r+m} n^m = 0.$

Caſus particularis, corollario hoc traditi, eximiæ in ſequentibus applicationes occurrent.

§. *s.* Eos inter caſus, quibus ad differentias conſtantes nunquam pervenitur, notandus inprimis eſt ille, quo ſeries propoſita eſt ſeries geometrica.

Theorema. Sit progreſſio geometrica quæcunque. Sumantur differentiæ omnium ordinum terminorum hujus progreſſionis: ſeries harum differentiarum pariter ſunt progreſſiones geometricæ, eundem, quem prima, exponentem habentes. Sumatur autem exceſſus, quo exponens iſte ſuperat unitatem, aut ab illa ſuperatur, prouti progreſſio eſt creſcens aut decreſcens; & ſumantur poteſtates hujus exceſſus, quarum exponens æqualis eſt ordini harum differentiarum: ſeries illæ erunt reſpective producta ex terminis primæ ſeriei per has poteſtates.

Sit $a^z, a^{2z}, a^{3z}, a^{4z}, a^{5z}, \ldots a^{(n-2)z}, a^{(n-1)z}, a^{nz}$ progreſſio geometrica, cujus exponens a^z.

Series differentiarum primarum eſt

$$a^{2z}-a^z,\ a^{3z}-a^{2z},\ a^{4z}-a^{3z},\ a^{5z}-a^{4z},\ \ldots\ a^{(n-1)z}-a^{(n-2)z},\ a^{nz}-a^{(n-1)z}$$

ſeu $(a^z-1)(a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ \ldots .\ a^{(n-2)z},\ a^{(n-1)z})$

Hinc ſeries differentiarum ſecundarum eſt

$$(a^z-1)^2(a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ \ldots .\ a^{(n-3)z},\ a^{(n-2)z})$$

Hinc rurſus ſeries differentiarum tertiarum eſt

$$(a^z-1)^3(a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ \ldots .\ a^{(n-4)z},\ a^{(n-3)z})$$

Item ſeries differentiarum quartarum eſt

$$(a^z-1)^4(a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ \ldots .\ a^{(n-5)z},\ a^{(n-4)z})$$

&c. &c. &c.

Generatim ſeries differentiarum m^{tarum} eſt

$$(a^z-1)^m(a^z,\ a^{2z},\ a^{3z},\ a^{4z},\ \ldots .\ a^{n-(m-1)z},\ a^{(n-m)z}).$$

§. *t.* Notari etiam merentur omnium ordinum differentiæ ſinuum & coſinuum arcuum in progreſſione arithmetica creſcentium aut decreſcentium; & inprimis (propter applicationes ſequentes) differentiæ ſinuum & coſinuum arcuum juxta ſeriem numerorum naturalium creſcentium.

Lemmata nota. 1°. Differentia ſinuum duorum arcuum æqualis eſt duplo producto coſinus dimidiæ ſummæ per ſinum dimidiæ differentiæ horum arcuum.

2°. Differentia coſinuum duorum arcuum æqualis eſt duplo producto ſinuum dimidiæ ſummæ & dimidiæ differentiæ horum arcuum.

Hoc eſt: 1°. $\text{ſin.}a - \text{ſin.}b = 2\text{coſ.}\frac{a+b}{2}\,\text{ſin.}\frac{a-b}{2}$

2°. $\text{coſ.}a - \text{coſ.}b = 2\text{ſin.}\frac{a+b}{2}\,\text{ſin.}\frac{b-a}{2}$.

Applicatio. Sint ſin. a, ſin. $2a$, ſin. $3a$, ſin. $4a$, ſin. $5a$, . . . ſin. na ſinus arcuum juxta arithmeticam numerorum naturalium progreſſionem creſcentium.

Series differentiarum primarum erit

$$2\text{ſin.}\tfrac{1}{2}a\,(\text{coſ.}\tfrac{3}{2}a,\ \text{coſ.}\tfrac{5}{2}a,\ \text{coſ.}\tfrac{7}{2}a,\ \text{coſ.}\tfrac{9}{2}a,\ \text{coſ.}\tfrac{11}{2}a,\ \text{coſ.}\tfrac{13}{2}a\ \ldots.)$$

Series differentiarum ſecundarum erit

$$-2^2\text{ſin.}^2\tfrac{1}{2}a\,(\text{ſin.}2a,\ \text{ſin.}3a,\ \text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a\ \ldots.)$$

Series differentiarum tertiarum

$$-2^3\text{ſin.}^3\tfrac{1}{2}a\,(\text{coſ.}\tfrac{5}{2}a,\ \text{coſ.}\tfrac{7}{2}a,\ \text{coſ.}\tfrac{9}{2}a,\ \text{coſ.}\tfrac{11}{2}a,\ \text{coſ.}\tfrac{13}{2}a,\ \text{coſ.}\tfrac{15}{2}a\ \ldots.)$$

Series differentiarum quartarum

$$+2^4\text{ſin.}^4\tfrac{1}{2}a\,(\text{ſin.}3a,\ \text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a,\ \text{ſin.}8a\ \ldots.)$$

Series differentiarum quintarum

$$+2^5\text{ſin.}^5\tfrac{1}{2}a\,(\text{coſ.}\tfrac{7}{2}a,\ \text{coſ.}\tfrac{9}{2}a,\ \text{coſ.}\tfrac{11}{2}a,\ \text{coſ.}\tfrac{13}{2}a,\ \text{coſ.}\tfrac{15}{2}a,\ \text{coſ.}\tfrac{17}{2}a\ \ldots.)$$

Series differentiarum ſextarum

$$-2^6\text{ſin.}^6\tfrac{1}{2}a\,(\text{ſin.}4a,\ \text{ſin.}5a,\ \text{ſin.}6a,\ \text{ſin.}7a,\ \text{ſin.}8a,\ \text{ſin.}9a\ \ldots.).$$

Generatim ſeries differentiarum ordinis paris $2m$ eſt

$$2^{2m}\text{ſin.}^{2m}\tfrac{1}{2}a\,(\text{ſin.}(m+1)a,\ \text{ſin.}(m+2)a,\ \text{ſin.}(m+3)a,\ \text{ſin.}(m+4)a,\ \text{ſin.}(m+5)a\ \ldots.)$$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{par}\\ \text{impar}\end{matrix}$.

Series differentiarum ordinis imparis $2m+1$ eſt

$$2^{2m+1}\text{ſin.}^{2m+1}\tfrac{1}{2}a\,(\text{coſ.}\frac{2m+3}{2}a,\ \text{coſ.}\frac{2m+5}{2}a,\ \text{coſ.}\frac{2m+7}{2}a,\ \text{coſ.}\frac{2m+9}{2}a,\ \text{coſ.}\frac{2m+11}{2}a\ \ldots.)$$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{o vel par}\\ \text{impar}\end{matrix}$.

2°. Sint

2°. Sint coſ.a, coſ.$2a$, coſ.$3a$, coſ.$4a$, coſ.$5a$, coſ.$6a$, ... coſ.na, coſinus arcuum juxta feriem numerorum naturalium creſcentium.

Series differentiarum primarum eſt

$-2\,\text{ſin.}\tfrac{1}{2}a\,(\text{ſin.}\tfrac{3}{2}a,\ \ \text{ſin.}\tfrac{5}{2}a,\ \ \text{ſin.}\tfrac{7}{2}a,\ \ \text{ſin.}\tfrac{9}{2}a,\ \ \text{ſin.}\tfrac{11}{2}a,\ \ \text{ſin.}\tfrac{13}{2}a\ \ldots)$

Series differentiarum ſecundarum eſt

$-2^2\,\text{ſin.}^2\tfrac{1}{2}a\,(\text{coſ.}2a,\ \ \text{coſ.}3a,\ \ \text{coſ.}4a,\ \ \text{coſ.}5a,\ \ \text{coſ.}6a,\ \ \text{coſ.}7a\ \ldots)$

Series differentiarum tertiarum eſt

$+2^3\,\text{ſin.}^3\tfrac{1}{2}a\,(\text{ſin.}\tfrac{5}{2}a,\ \ \text{ſin.}\tfrac{7}{2}a,\ \ \text{ſin.}\tfrac{9}{2}a,\ \ \text{ſin.}\tfrac{11}{2}a,\ \ \text{ſin.}\tfrac{13}{2}a,\ \ \text{ſin.}\tfrac{15}{2}a\ \ldots)$

Series differentiarum quartarum eſt

$2^4\,\text{ſin.}^4\tfrac{1}{2}a\,(\text{coſ.}3a,\ \ \text{coſ.}4a,\ \ \text{coſ.}5a,\ \ \text{coſ.}6a,\ \ \text{coſ.}7a,\ \ \text{coſ.}8a\ \ldots)$

Series differentiarum quintarum eſt

$-2^5\,\text{ſin.}^5\tfrac{1}{2}a\,(\text{ſin.}\tfrac{7}{2}a,\ \ \text{ſin.}\tfrac{9}{2}a,\ \ \text{ſin.}\tfrac{11}{2}a,\ \ \text{ſin.}\tfrac{13}{2}a,\ \ \text{ſin.}\tfrac{15}{2}a,\ \ \text{ſin.}\tfrac{17}{2}a\ \ldots)$

Series differentiarum ſextarum eſt

$-2^6\,\text{ſin.}^6\tfrac{1}{2}a\,(\text{coſ.}4a,\ \ \text{coſ.}5a,\ \ \text{coſ.}6a,\ \ \text{coſ.}7a,\ \ \text{coſ.}8a,\ \ \text{coſ.}9a\ \ldots)$

Generatim ſeries differentiarum ordinis paris $2m$ eſt

$2^{2m}\,\text{ſin.}^{2m}\tfrac{1}{2}a\,(\text{coſ.}(m+1)a,\ \ \text{coſ.}(m+2)a,\ \ \text{coſ.}(m+3)a,\ \ \text{coſ.}(m+4)a,\ \ \text{coſ.}(m+5)a\ \ldots)$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{par}\\ \text{impar}\end{matrix}$.

Et ſeries differentiarum ordinis imparis $2m+1$ eſt

$2^{2m+1}\,\text{ſin.}^{2m+1}\tfrac{1}{2}a\,(\text{ſin.}\frac{2m+3}{2}a,\ \text{ſin.}\frac{2m+5}{2}a,\ \text{ſin.}\frac{2m+7}{2}a,\ \text{ſin.}\frac{2m+9}{2}a,\ \text{ſin.}\frac{2m+11}{2}a\ \ldots)$

cum alterutro ſigno $\pm$, prouti m eſt $\begin{matrix}\text{impar}\\ \text{o vel par}\end{matrix}$.

Scholium. Omiſſo factore $2^m\text{ſin.}^m\tfrac{1}{2}a$, ſeries harum differentiarum ſucceſſivarum, pariter atque ipſæ ſeries primitivæ, periodice in ſe redeunt, aut ſunt ſemper diverſæ; prouti arcus a & dimidia circumferentia commenſurabiles, aut incommenſurabiles ſunt.

Monenda de compendioſo differentiarum ſymbolismo.

In ſequentibus, compendii & majoris impreſſionis facilitatis cauſa, differentiæ ordinum ſucceſſivorum potentiarum numerorum naturalium, quarum exponens m, plerumque deſignabuntur, ut ſequitur:

 Diffe-

Differentiæ		Deſignatio compendioſa
$2^{dæ}$	$n^m - 2(n-1)^m + (n-2)^m$	$\Delta''(n^m \dots (n-2)^m)$
$3^{tiæ}$	$n^m - 3(n-1)^m + 3(n-2)^m - (n-3)^m$	$\Delta'''(n^m \dots (n-3)^m)$
$4^{tæ}$	$n^m - 4(n-1)^m + 6(n-2)^m - 4(n-3)^m + (n-4)^m$	$\Delta^{iv}(n^m \dots (n-4)^m)$
$5^{tæ}$	$n^m - 5(n-1)^m + 10(n-2)^m - 10(n-3)^m + 5(n-4)^m - (n-5)^m$	$\Delta^{v}(n^m \dots (n-5)^m)$.
⋮		⋮
$p^{tæ}$	$n^m - \frac{p}{1}(n-1)^m + \frac{p}{1}\cdot\frac{p-1}{2}(n-2)^m \dots \mp \frac{p}{1}(n-(p-1))^m \pm (n-p)^m$	$\Delta^{p}(n^m \dots (n-p)^m)$.

Caput C.

De ſummis poteſtatum numerorum æquidifferentium.

Summæ poteſtatum numerorum æquidifferentium (quos inter ſeries numerorum naturalium primum tenet locum) adeo frequenter calculos ſuperiores ingrediuntur, ut ſtrictim de illis in hac introductione dicere conſultum duxerim.

Plures mathematici variis modis eas indagarunt atque expoſuerunt. Cum autem ſcopus in hac tractatione mihi præcipue propoſitus ſit illius ad limites dictarum ſummarum applicatio: nulla methodus potior & huic fini melius accommodata ſe mihi obtulit, quam ea, qua uſus eſt ſagaciſſimus Pascal in eleganti opuſculo, cui titulus: *Poteſtatum numericarum ſumma* (quod pag. 34 ſqq. inſertum eſt libro ipſius inſcripto: *Traité du triangle arithmetique*. Paris 1665.); & qua poteſtatum cujuslibet ordinis ſumma ad ſummas potentiarum ordinum inferiorum reducitur.

§. *u*. *Lemma.* Sint a & $a+d$ duo numeri ad potentiam quamlibet evecti, cujus exponens r. Differentia $(a+d)^r - a^r$ harum poteſtatum (per §. *e*.) erit

$$\frac{r}{1}a^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}a^{r-2}d^2 + \frac{r}{1}\cdots\frac{r-2}{3}a^{r-3}d^3 + \frac{r}{1}\cdots\frac{r-3}{4}a^{r-4}d^4 + \dots + \frac{r}{1}\cdot\frac{r-1}{2}a^2d^{r-2} + \frac{r}{1}ad^{r-1} + d^r.$$

§. *v*. Sit ſeries terminorum æquidifferentium, quorum minimus ſit a, differentia communis d, & numerus n.

Sit (compendii cauſa) $\int na^r$ ſumma poteſtatum, quarum exponens r, terminorum illorum inde a primo a usque ad n^{tum} $a+(n-1)d$. Accipiatur etiam eadem poteſtas termini $a+nd$ ultimum ſequentis.

Sumtis differentiis poteſtatum ejusdem exponentis quorumlibet duorum terminorum contiguorum, erit (§. *u*.)

$(a+d)^r$

$$(a+d)^r - a^r = \frac{r}{1}a^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}a^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}a^{r-3}d^3 \;\ldots\ldots + \frac{r}{1}ad^{r-1} + d^r$$

$$(a+2d)^r - (a+d)^r = \frac{r}{1}(a+d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+d)^{r-3}d^3 \ldots + \frac{r}{1}(a+d)d^{r-1} + d^r$$

$$(a+3d)^r - (a+2d)^r = \frac{r}{1}(a+2d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+2d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+2d)^{r-3}d^3 \ldots + \frac{r}{1}(a+2d)d^{r-1} + d^r$$

$$(a+4d)^r - (a+3d)^r = \frac{r}{1}(a+3d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+3d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-1}{2}(a+3d)^{r-3}d^3 \ldots + \frac{r}{1}(a+3d)d^{r-1} + d^r$$

$$\vdots$$

$$(a+(n-2)d)^r - (a+(n-3)d)^r = \frac{r}{1}(a+(n-3)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-3)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-3)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-3)d)d^{r-1} + d^r$$

$$(a+(n-1)d)^r - (a+(n-2)d)^r = \frac{r}{1}(a+(n-2)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-2)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-2)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-2)d)d^{r-1} + d^r$$

$$(a+nd)^r - (a+(n-1)d)^r = \frac{r}{1}(a+(n-1)d)^{r-1}d + \frac{r}{1}\cdot\frac{r-1}{2}(a+(n-1)d)^{r-2}d^2 + \frac{r}{1}\ldots\frac{r-2}{3}(a+(n-1)d)^{r-3}d^3 \ldots + \frac{r}{1}(a+(n-1)d)d^{r-1} + d^r.$$

Summa omnium priorum membrorum harum æquationum eſt $(a+nd)^r - a^r$.

Et ſumma omnium poſteriorum membrorum eſt

$$\frac{r}{1}d\int.na^{r-1} + \frac{r}{1}\cdot\frac{r-1}{2}d^2\int.na^{r-2} + \frac{r}{1}\ldots\frac{r-2}{3}d^3\int.na^{r-3} + \frac{r}{1}\ldots\frac{r-3}{4}d^4\int.na^{r-4} + \ldots\ldots + \frac{r}{1}d^{r-1}\int.na + nd^r.$$

Hinc $\frac{r}{1}d\int.na^{r-1} = (a+nd)^r - a^r - \left(\frac{r}{1}\cdot\frac{r-1}{2}d^2\int.na^{r-2} + \frac{r}{1}\ldots\frac{r-2}{3}d^3\int.na^{r-3} + \ldots\ldots + \frac{r}{1}d^{r-1}\int.na + nd^r\right)$

unde $\frac{r+1}{1}d\int.na^r = (a+nd)^{r+1} - a^{r+1} - \left(\frac{r+1}{1}\cdot\frac{r}{2}d^2\int.na^{r-1} + \frac{r+1}{1}\ldots\frac{r-1}{3}d^3\int.na^{r-2} + \ldots\ldots + \frac{r+1}{1}d^r\int.na + nd^{r+1}\right)$.

Et ſic ſumma poteſtatum cujuslibet ordinis per legem omnino regularem ad ſummas poteſtatum omnium ordinum inferiorum reducitur.

§. x. *Exemplum.* Series numerorum æquidifferentium ſit ſeries numerorum naturalium ab unitate usque ad n; ubi $a=1$, $d=1$. Et ſit $\int n^r$ ſumma poteſtatum ordinis r primorum n numerorum naturalium. Fit

$$\frac{r+1}{1}\int.n^r = (n+1)^{r+1} - 1 - \left(\frac{r+1}{1}\cdot\frac{r}{2}\int.n^{r-1} + \frac{r+1}{1}\ldots\frac{r-1}{3}\int.n^{r-2} + \frac{r+1}{1}\ldots\frac{r-2}{4}\int.n^{r-3} + \frac{r+1}{1}\ldots\frac{r-3}{5}\int.n^{r-4} \ldots\ldots + \frac{r+1}{1}\int.n + n\right).$$

Sit $r=1$. Fit $2\int.n = (n+1)^2 - 1 - n = n.\overline{n+1}$

$r=2$ $\quad 3\int.n^2 = (n+1)^3 - 1 - \left(\frac{3.2}{1.2}\int.n + n\right) = \frac{n.\overline{n+1}.\overline{2n+1}}{1.2}$

$r=3$

$$r=3 \qquad 4\,\text{ſ}.n^3=(n+1)^4-1-\left(\frac{4.3}{1.2}\text{ſ}.n^2+\frac{4.3.2}{1.2.3}\text{ſ}.n+n\right)=n^2.(n+1)^2$$

$$r=4 \qquad 5\,\text{ſ}.n^4=(n+1)^5-1-\left(\frac{5.4}{1.2}\text{ſ}.n^3+\frac{5\cdots3}{1\ldots3}\text{ſ}.n^2+\frac{5\cdots2}{1\ldots4}\text{ſ}.n+n\right)$$

$$r=5 \qquad 6\,\text{ſ}.n^5=(n+1)^6-1-\left(\frac{6.5}{1.2}\text{ſ}.n^4+\frac{6\ldots4}{1\ldots3}\text{ſ}.n^3+\frac{6\ldots3}{1\ldots4}\text{ſ}.n^2+\frac{6\ldots2}{1\ldots5}\text{ſ}.n+n\right)$$

$$r=6 \qquad 7\,\text{ſ}.n^6=(n+1)^7-1-\left(\frac{7.6}{1.2}\text{ſ}.n^5+\frac{7\cdots5}{1\ldots3}\text{ſ}.n^4+\frac{7\cdots4}{1\ldots4}\text{ſ}.n^3+\frac{7\cdots3}{1\ldots5}\text{ſ}.n^2+\frac{7\cdots2}{1\ldots6}\text{ſ}.n+n\right).$$

§. *y*. Ad ſcopum noſtrum pertinet obſervare, quod

$\text{ſ}n = An^2+Bn$ - - - - - - - - - ubi $A=\frac{1}{2}$

$\text{ſ}n^2 = An^3+Bn^2+Cn$ - - - - - - - - ubi $A=\frac{1}{3}$

$\text{ſ}n^3 = An^4+Bn^3+Cn^2+Dn$ - - - - - - ubi $A=\frac{1}{4}$

$\text{ſ}n^4 = An^5+Bn^4+Cn^3+Dn^2+En$ - - - - - ubi $A=\frac{1}{5}$

$\text{ſ}n^5 = An^6+Bn^5+Cn^4+Dn^3+En^2+Fn$ - - - - ubi $A=\frac{1}{6}$

$\text{ſ}n^6 = An^7+Bn^6+Cn^5+Dn^4+En^3+Fn^2+Gn$ - - - ubi $A=\frac{1}{7}$

⋮

$\text{ſ}n^r = An^{r+1}+Bn^r+Cn^{r-1}+Dn^{r-2}+En^{r-3}+Fn^{r-4}+Gn^{r-5}\ldots+Ln^2+Mn$, ubi $A=\frac{1}{r+1}$.

Quod ad reliquos coëfficientes attinet: *B* eſt conſtans, nempe $=\frac{1}{2}$. Cæteri autem *C*, *D*, *E* .... *L*, *M* pendent ab exponente variabili *r*, juxta legem quamdam, cujus evolutio ad ſcopum præſentem non pertinet; & de qua videatur inter alios Jac. Bernoulli *Ars conjectandi* pag. 97.

Caput D.

De ſinibus et coſinibus arcuum multiplorum, et de ſinuum in factores reſolutione.

§. *z*. *Lemma notum.* 1°. $\text{ſin.}(a+b) = \text{ſin.}\,a\,\text{coſ.}\,b+\text{coſ.}\,a\,\text{ſin.}\,b$

2°. $\text{coſ.}(a+b) = \text{coſ.}\,a\,\text{coſ.}\,b-\text{ſin.}\,a\,\text{ſin.}\,b$.

§. *aa*. Ex his duabus formulis deduci poſſunt expreſſiones ſinuum & coſinuum arcuum, qui multipli ſunt alicujus arcus propoſiti. Scilicet

1°. coſ.

1°. coſ.$2a$(=coſ.$(a+a)$) = coſ.^2a − ſin.^2a

ſin.$2a$(=ſin.$(a+a)$) = 2ſin.a coſ.a.

2°. coſ.$3a$(=coſ.$(2a+a)$)=coſ.$2a$coſ.a−ſin.$2a$ſin.a
= coſ.^3a
− 3coſ.a ſin.^2a

ſin.$3a$(=ſin.$(2a+a)$)=ſin.$2a$coſ.a + coſ.$2a$ſin.a
= 3ſin.a coſ.^2a
− ſin.^3a.

3°. coſ.$4a$(=coſ.$(3a+a)$)=coſ.$3a$coſ.a−ſin.$3a$ſin.a
= coſ.^4a
− 6coſ.^2a ſin.^2a
+ ſin.^4a

ſin.$4a$(=ſin.$(3a+a)$)=ſin.$3a$coſ.a + coſ.$3a$ſin.a
= 4coſ.^3a ſin.a
− 4coſ.a ſin.^3a.

4°. coſ.$5a$(=coſ.$(4a+a)$)=coſ.$4a$coſ.a−ſin.$4a$ſin.a
= coſ.^5a
− 10coſ.^3a ſin.^2a
+ 5coſ.a ſin.^4a

ſin.$5a$(=ſin.$(4a+a)$)=ſin.$4a$coſ.a + coſ.$4a$ſin.a
= 5coſ.^4a ſin.a
− 10coſ.^2a ſin.^3a
+ ſin.^5a.

5°. coſ.$6a$(=coſ.$(5a+a)$)=coſ.$5a$coſ.a−ſin.$5a$ſin.a
= coſ.^6a
− 15coſ.^4a ſin.^2a
+ 15coſ.^2a ſin.^4a
− ſin.^6a.

ſin.$6a$(=ſin.$(5a+a)$)=ſin.$5a$coſ.a + coſ.$5a$ſin.a
= 6coſ.^5a ſin.a
− 20coſ.^3a ſin.^3a
+ 6coſ.a ſin.^5a

Exempla hæc ſufficiunt ad indicandam legem harum expreſſionum, qua fit

$$\text{coſ.}2ma = \text{coſ.}^{2m}a - \frac{2m}{1}\cdot\frac{2m-1}{2}\text{coſ.}^{2m-2}a\,\text{ſin.}^2a + \frac{2m}{1}\cdots\frac{2m-3}{4}\text{coſ.}^{2m-4}a\,\text{ſin.}^4a - \frac{2m}{1}\cdots\frac{2m-5}{6}\text{coſ.}^{2m-6}a\,\text{ſin.}^6a + \cdots \pm \frac{2m}{1}\cdot\frac{2m-1}{2}\text{coſ.}^2\,\text{ſin.}^{2m-2}a \mp \text{ſin.}^{2m}a$$

$$\text{ſin.}2ma = \frac{2m}{1}\text{coſ.}^{2m-1}a\,\text{ſin.}a - \frac{2m}{1}\cdots\frac{2m-2}{3}\text{coſ.}^{2m-3}a\,\text{ſin.}^3a + \frac{2m}{1}\cdots\frac{2m-4}{5}\text{coſ.}^{2m-5}a\,\text{ſin.}^5a - \frac{2m}{1}\cdots\frac{2m-6}{7}\text{coſ.}^{2m-7}a\,\text{ſin.}^7a + \cdots \mp \frac{2m}{1}\cdots\frac{2m-2}{3}\text{coſ.}^3a\,\text{ſin.}^{2m-3}a \pm \frac{2m}{1}\text{coſ.}a\,\text{ſin.}^{2m-1}a$$

$$\text{coſ.}(2m+1)a = \text{coſ.}^{2m+1}a - \frac{2m+1}{1}\cdot\frac{2m}{2}\text{coſ.}^{2m-1}a\,\text{ſin.}^{2}a + \frac{2m+1}{1}\cdots\frac{2m-2}{4}\text{coſ.}^{2m-3}a\,\text{ſin.}^{4}a - \frac{2m+1}{1}\cdots\frac{2m-4}{6}\text{coſ.}^{2m-5}a\,\text{ſin.}^{6}a + \cdots \pm \frac{2m+1}{1}\cdots\frac{2m-1}{3}\text{coſ.}^{3}a\,\text{ſin.}^{2m-2}a \mp \frac{2m+1}{1}\text{coſ.}a\,\text{ſin.}^{2m}a$$

$$\text{ſin.}(2m+1)a = \frac{2m+1}{1}\text{coſ.}^{2m}a\,\text{ſin.}a - \frac{2m+1}{1}\cdots\frac{2m-1}{3}\text{coſ.}^{2m-2}a\,\text{ſin.}^{3}a + \frac{2m+1}{1}\cdots\frac{2m-3}{5}\text{coſ.}^{2m-4}a\,\text{ſin.}^{5}a - \frac{2m+1}{1}\cdots\frac{2m-5}{6}\text{coſ.}^{2m-6}a\,\text{ſin.}^{7}a + \cdots \mp \frac{2m+1}{1}\cdot\frac{2m}{2}\text{coſ.}^{2}a\,\text{ſin.}^{2m-1}a \pm \text{ſin.}^{2m+1}a.$$

Demonſtratur autem modo mathematicis familiari: quod, ſi hæc lex obtineat pro exponente quocunque propoſito, eadem etiam valeat pro exponente unitate majori. Sed lex locum habet pro exponentibus 2, 3 . . . 6: ergo & locum habet pro exponente ſequente 7; inde pro exponente 8; & ſic deinceps.

§. *ab.* Termini igitur, quibus expreſſio coſinus arcus m^{pli} arcus a conſtat, ſunt termini alterni binomii $(\text{coſ.}z+\text{ſin.}z)^m$; ſed alternis vicibus, a primo inde, ſignis + & — affecti: & termini, quibus conſtat expreſſio ſinus arcus ejusdem m^{pli} ſunt reliqui termini alterni ejusdem binomii, alternis vicibus a ſecundo inde ſignis + & — pariter affecti.

Quoniam autem dimidia ſumma potentiarum ejusdem exponentis m ſummæ $a+b$ & differentiæ $a-b$ duarum quantitatum realium a & b conſtat terminis alternis binomii $(a+b)^m$, a primo inde, iisque ſigno + affectis; dum dimidia differentia harum poteſtatum continet reliquos terminos alternos etiam poſitivos: ut expreſſio coſinuum arcuum multiplorum reduci poſſit ad ſummam poteſtatum ſimilium ſummæ & differentiæ coſinus & ſinus arcus ſimplicis; ſinus hic affici debet coëfficiente tali, ut potentiæ ejus impariter pares ſigno — afficiantur, poteſtates vero pariter pares maneant poſitivæ. Quod fiet, ſi loco $\text{coſ.}z+\text{ſin.}z$ ſubſtituatur $\text{coſ.}z+\text{ſin.}z\sqrt{-1}$; unde ad hanc deducimur expreſſionem $\text{coſ.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2}$.

Evo-

Evoluta autem duarum poteſtatum $(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m$, $(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$ differentia, omnes ejus termini afficiuntur coëfficiente imaginario $\sqrt{-1}$; unde, ut expreſſio maneat realis, inferimus $\text{ſin.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}}$.

Quodſi enim expreſſiones hæ veræ ſint pro exponente quodam m, eædem etiam valent pro exponente unitate majore $m+1$.

Etenim

$$\text{coſ.}(m+1)z = \text{coſ.}mz\,\text{coſ.}z-\text{ſin.}mz\,\text{ſin.}z$$

$$= \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2}\times\frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})}{2}$$

$$-\frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}}\times\frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})}{2\sqrt{-1}}$$

Quæ ſumma reducitur ad $\frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^{m+1}+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^{m+1}}{2}$, uti propoſitum.

Eodemque modo fit $\text{ſin.}(m+1)z = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^{m+1}-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^{m+1}}{2\sqrt{-1}}$

Obſervatio. Expreſſiones hæ imaginariæ (quæ in calculo inprimis integrali, & in ſerierum ſummatione frequentiſſime uſurpantur) mera ſunt ſigna, majoris calculi facilitatis gratia ab mathematicis introducta. Partes earum $(\text{coſ.}z \pm \text{ſin.}z\sqrt{-1})^m$ ſeorſim ſumtæ omni ſenſu carent; & operationes his ſymbolis juxta regulas calculi litteralis ad eas extenſas inſtitutæ eatenus tantum quantitates reales ultimo exhibent, quatenus termini imaginarii, quos expreſſiones illæ continent, ſeſe mutuo deſtruunt. Scilicet æquatio $2a = (a+d)+(a-d)$ ſemper obtinet, quidquid ſit d; ideoque etiam ſi d involvat ſignum impoſſibilitatis, quod utramque expreſſionem $a+d$, $a-d$ pariter impoſſibilem reddit.

§. ac. Quoniam $\text{coſ.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2}$

et $\text{ſin.}mz = \frac{(\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m}{2\sqrt{-1}}$

erit $2\,\text{coſ.}mz = (\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m+(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$

et $2\,\text{ſin.}mz\sqrt{-1} = (\text{coſ.}z+\text{ſin.}z\sqrt{-1})^m-(\text{coſ.}z-\text{ſin.}z\sqrt{-1})^m$

Unde fit $(\cos z+\sin z\sqrt{-1})^m=\cos mz+\sin mz\sqrt{-1}$

et $(\cos z-\sin z\sqrt{-1})^m=\cos mz-\sin mz\sqrt{-1}$

Hinc $\cos z+\sin z\sqrt{-1} \quad =(\cos mz+\sin mz\sqrt{-1})^{\frac{1}{m}}$

et $\cos z-\sin z\sqrt{-1} \quad =(\cos mz-\sin mz\sqrt{-1})^{\frac{1}{m}}$

Unde $$\cos z \quad =\frac{(\cos mz+\sin mz\sqrt{-1})^{\frac{1}{m}}+(\cos mz-\sin mz\sqrt{-1})^{\frac{1}{m}}}{2}$$

$$\sin z \quad =\frac{(\cos mz+\sin mz\sqrt{-1})^{\frac{1}{m}}-(\cos mz-\sin mz\sqrt{-1})^{\frac{1}{m}}}{2\sqrt{-1}}.$$

Proinde eadem lex, quæ valet pro ſinibus & coſinibus arcuum multiplorum, pariter valet pro ſinibus & coſinibus arcuum ſubmultiplorum.

§. *ad. Theorema Coteſianum* (a ſagaciſſimo ejus auctore denominatum) illud eſt, quo demonſtratur, functiones utriusque formæ $a^n \pm b^n$ in factores binomios aut trinomios reſolvi modo ſequenti:

1°. $$\begin{aligned} a^{2n}+b^{2n} &= (aa-2ab\cos\tfrac{1}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{3}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{5}{2n}\pi+bb)\\ &\quad\vdots \qquad\qquad \vdots\\ &\times(aa-2ab\cos\tfrac{2n-3}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{2n-1}{2n}\pi+bb) \end{aligned}$$

2°. $$\begin{aligned} a^{2n+1}+b^{2n+1} &= (a+b)\times(aa+2ab\cos\tfrac{2}{2n+1}\pi+bb)\\ &\times(aa+2ab\cos\tfrac{4}{2n+1}\pi+bb)\\ &\times(aa+2ab\cos\tfrac{6}{2n+1}\pi+bb)\\ &\quad\vdots \qquad\qquad \vdots\\ &\times(aa+2ab\cos\tfrac{2n-2}{2n+1}\pi+bb)\\ &\times(aa+2ab\cos\tfrac{2n}{2n+1}\pi+bb) \end{aligned}$$

3°. $$\begin{aligned} a^{2n}-b^{2n} &= (aa-bb)\times(aa-2ab\cos\tfrac{2}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{4}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{6}{2n}\pi+bb)\\ &\quad\vdots \qquad\qquad \vdots\\ &\times(aa-2ab\cos\tfrac{2n-4}{2n}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{2n-2}{2n}\pi+bb) \end{aligned}$$

4°. $$\begin{aligned} a^{2n+1}-b^{2n+1} &= (a-b)\times(aa-2ab\cos\tfrac{2}{2n+1}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{4}{2n+1}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{6}{2n+1}\pi+bb)\\ &\quad\vdots \qquad\qquad \vdots\\ &\times(aa-2ab\cos\tfrac{2n-2}{2n+1}\pi+bb)\\ &\times(aa-2ab\cos\tfrac{2n}{2n+1}\pi+bb). \end{aligned}$$

Theo-

Theoremati huic demonſtrando (quod a variis auctoribus præſtitum eſt) non immoror; quamvis talis ejus demonſtrationis compotem me eſſe credam, quæ ſimplicitate ſua ſeſe commendat. Pergo ideo ad illius applicationem, quæ deinceps utilis erit.

§. *ae.* 1°. In prima formula fiat $a = b = 1$:

$$\text{fit}\quad 2 = 2(1-\text{coſ.}\tfrac{1}{2n}\pi)$$
$$\times 2(1-\text{coſ.}\tfrac{3}{2n}\pi)$$
$$\times 2(1-\text{coſ.}\tfrac{5}{2n}\pi)$$
$$\vdots \qquad \vdots$$
$$\times 2(1-\text{coſ.}\tfrac{2n-3}{2n}\pi)$$
$$\times 2(1-\text{coſ.}\tfrac{2n-1}{2n}\pi)$$
$$= 4^n\,\text{ſin.}^2\tfrac{1}{2n}p\;\text{ſin.}^2\tfrac{3}{2n}p\;\text{ſin.}^2\tfrac{5}{2n}p\ldots\text{ſin.}^2\tfrac{2n-3}{2n}p\;\text{ſin.}^2\tfrac{2n-1}{2n}p$$

Unde $\sqrt{2} = 2^n\,\text{ſin.}\tfrac{1}{2n}p\;\text{ſin.}\tfrac{3}{2n}p\;\text{ſin.}\tfrac{5}{2n}p\ldots\text{ſin.}\tfrac{2n-3}{2n}p\;\text{ſin.}\tfrac{2n-1}{2n}p.$

2°. Ex ſecunda formula fit pariter

$$\sqrt{2} = 2^n\,\text{ſin.}\tfrac{2}{2n+1}p\;\text{ſin.}\tfrac{4}{2n+1}p\;\text{ſin.}\tfrac{6}{2n+1}p\;.\;.\;.\;\text{ſin.}\tfrac{2n-2}{2n+1}p\;\text{ſin.}\tfrac{2n}{2n+1}p.$$

3°. $a^{2n}-b^{2n} = (aa-bb)(a^{2n-2}+a^{2n-4}bb+a^{2n-6}b^4+\ldots+a^2b^{2n-4}+b^{2n-2})$

Hinc $a^{2n-2}+a^{2n-4}bb+a^{2n-6}b^4+\ldots+a^2b^{2n-4}+b^{2n-2} = (aa-2ab\,\text{coſ.}\tfrac{1}{n}\pi+bb)$

$$\times(aa-2ab\,\text{coſ.}\tfrac{2}{n}\pi+bb)$$
$$\times(aa-2ab\,\text{coſ.}\tfrac{3}{n}\pi+bb)$$
$$\vdots \qquad \vdots$$
$$\times(aa-2ab\,\text{coſ.}\tfrac{n-2}{n}\pi+bb)$$
$$\times(aa-2ab\,\text{coſ.}\tfrac{n-1}{n}\pi+bb).$$

Hinc factis $a=b=1$: $n = 4^{n-1}\text{ſin.}^2\tfrac{1}{n}p\;\text{ſin.}^2\tfrac{2}{n}p\;\text{ſin.}^2\tfrac{3}{n}p\;.\;.\;.\;\text{ſin.}^2\tfrac{n-2}{n}p\;\text{ſin.}^2\tfrac{n-1}{n}p$

et $\sqrt{n} = 2^{n-1}\,\text{ſin.}\tfrac{1}{n}p\;\text{ſin.}\tfrac{2}{n}p\;\text{ſin.}\tfrac{3}{n}p\;.\;.\;.\;\text{ſin.}\tfrac{n-2}{n}p\;\text{ſin.}\tfrac{n-1}{n}p.$

4°. Ex

4°. Ex quarta formula fit pariter

$$\mathcal{V}(2n+1) = 2^n \,\text{ſin.}\, \frac{2}{2n+1} p \,\text{ſin.}\, \frac{4}{2n+1} p \,\text{ſin.}\, \frac{6}{2n+1} p \dots \text{ſin.}\, \frac{2n-2}{2n+1} p \,\text{ſin.}\, \frac{2n}{2n+1} p.$$

§. *af.* Poſt *Coteſium* theorema ipſius fuit amplificatum, & formula quoque $a^{2n} \pm 2a^n b^n \,\text{coſ.}\varphi + b^{2n}$ reſoluta in factores, ut ſequitur, prouti n numerus eſt par aut impar:

1°. $a^{4n} - 2a^{2n}b^{2n}\,\text{coſ.}\,2\varphi + b^{4n}$

$$\begin{aligned}
&= \left(aa - 2ab \,\text{coſ.}\, \frac{2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2\pi - 2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2\pi + 2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{4\pi - 2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{4\pi + 2\varphi}{2n} + bb\right) \\
&\quad\vdots \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{(2n-2)\pi - 2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{(2n-2)\pi + 2\varphi}{2n} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2n\pi - 2\varphi}{2n} + bb\right)
\end{aligned}$$

2° $a^{4n+2} - 2a^{n+1}b^{n+1}\,\text{coſ.}\,2\varphi + b^{4n+2}$

$$\begin{aligned}
&= \left(aa - 2ab \,\text{coſ.}\, \frac{2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2\pi - 2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2\pi + 2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{4\pi - 2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{4\pi + 2\varphi}{2n+1} + bb\right) \\
&\quad\vdots \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{(2n-2)\pi + 2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2n\pi - 2\varphi}{2n+1} + bb\right) \\
&\times \left(aa - 2ab \,\text{coſ.}\, \frac{2n\pi + 2\varphi}{2n+1} + bb\right).
\end{aligned}$$

§. *ag.* Hinc factis $a = b = 1$, erit

1°. $$4\,\text{ſin.}^2\varphi = 4^{2n}\,\text{ſin.}^2\frac{\varphi}{2n}\,\text{ſin.}^2\frac{\pi-\varphi}{2n}\,\text{ſin.}^2\frac{\pi+\varphi}{2n}\,\text{ſin.}^2\frac{2\pi-\varphi}{2n}\,\text{ſin.}^2\frac{2\pi+\varphi}{2n}\dots\text{ſin.}^2\frac{(n-1)\pi-\varphi}{2n}\,\text{ſin.}^2\frac{(n-1)\pi+\varphi}{2n}\,\text{ſin.}^2\frac{n\pi-\varphi}{2n}$$

et $$\text{ſin.}\varphi = 2^{2n-1}\,\text{ſin.}\frac{\varphi}{2n}\,\text{ſin.}\frac{\pi-\varphi}{2n}\,\text{ſin.}\frac{\pi+\varphi}{2n}\,\text{ſin.}\frac{2\pi-\varphi}{2n}\,\text{ſin.}\frac{2\pi+\varphi}{2n}\dots\text{ſin.}\frac{(n-1)\pi-\varphi}{2n}\,\text{ſin.}\frac{(n-1)\pi+\varphi}{2n}\,\text{ſin.}\frac{n\pi-\varphi}{2n}$$

2°. $$4\,\text{ſin.}^2\varphi = 4^{2n+1}\,\text{ſin.}^2\frac{\varphi}{2n+1}\,\text{ſin.}^2\frac{\pi-\varphi}{2n+1}\,\text{ſin.}^2\frac{\pi+\varphi}{2n+1}\,\text{ſin.}^2\frac{2\pi-\varphi}{2n+1}\,\text{ſin.}^2\frac{2\pi+\varphi}{2n+1}\dots\text{ſin.}^2\frac{n\pi-\varphi}{2n+1}\,\text{ſin.}^2\frac{n\pi+\varphi}{2n+1}$$

et $$\text{ſin.}\varphi = 2^{2n}\,\text{ſin.}\frac{\varphi}{2n+1}\,\text{ſin.}\frac{\pi-\varphi}{2n+1}\,\text{ſin.}\frac{\pi+\varphi}{2n+1}\,\text{ſin.}\frac{2\pi-\varphi}{2n+1}\,\text{ſin.}\frac{2\pi+\varphi}{2n+1}\dots\text{ſin.}\frac{n\pi-\varphi}{2n+1}\,\text{ſin.}\frac{n\pi+\varphi}{2n+1}.$$

Caput Primum.

De Limitibus Quantitatum et Rationum, ſeu de Methodo Exhauſtionis.

§. 1.

Definitiones. 1°. Si quantitas mutabilis ſemper minor fuerit quantitate datâ (mutabili ſcilicet homogenea, quod deinceps ſemper ſubintelligatur); ſed ita augeri poterit, ut major fiat quacunque quantitate datâ, quæ minor eſt prima quantitate datâ (& tam cum hac quam cum mutabili homogenea, quod pariter deinceps ſemper ſubintelligatur): hæc prima quantitas data, dicitur *Limes Quantitatis mutabilis creſcentis.*

2°. Et ſi quantitas mutabilis ſemper major fuerit quantitate datâ, ſed ita minui poterit, ut minor fiat quacunque quantitate datâ, quæ major eſt prima quantitate datâ: hæc prima quantitas data dicitur *Limes Quantitatis mutabilis decreſcentis.*

3°. Si ratio mutabilis ſemper minor fuerit quam ratio data; ſed ita augeri poterit, ut major fiat ratione quacunque datâ, quæ minor eſt ratione prima datâ: hæc prima ratio data dicitur *Limes Rationis mutabilis creſcentis.*

4°. Si ratio mutabilis ſemper major fuerit quam ratio data; ſed ita minui poterit, ut minor fiat ratione quacunque datâ, quæ major eſt ratione prima datâ: hæc prima ratio data dicitur *Limes Rationis mutabilis decreſcentis.* (a)

E re eſſe cenſeo definitiones has nonnullis exemplis illuſtrare.

Sit progreſſio geometrica decreſcens: $1, p, p^2, p^3, p^4 \dots p^{n-1}$; in qua proinde p eſt minor unitate. Summa n primorum terminorum hujus progreſ-

(a) Definitiones hæ, pariter ac propoſitiones nonnullæ hoc capite expoſitæ, deſumptæ ſunt ex Opuſculo R. Simson, inſcripto: *De Limitibus Quantitatum & Rationum Fragmentum;* quod reperitur inter Opera poſthuma inſignis illius Geometræ, impenſis Viri eruditiſſimi, & ſcientiarum mathematicarum fautoris generoſiſſimi Comitis Stanhope in lucem edita: Glasguæ 1776.

greſſionis eſt: $\frac{1-p^n}{1-p} = \frac{1}{1-p} - \frac{p^n}{1-p}$. Ideo ſumma quotcunque terminorum hujus progreſſionis minor eſt quantitate $\frac{1}{1-p}$. Atqui: auctô n, quantitas p^n, & ideo etiam quantitas $\frac{p^n}{1-p}$, poteſt fieri minor quacunque quantitate datâ; ideo quantitas $\frac{1}{1-p} - \frac{p^n}{1-p}$ poteſt fieri major quacunque quantitate datâ, quæ minor eſt quantitate datâ $\frac{1}{1-p}$. Hinc quantitas $\frac{1}{1-p}$ dicitur limes valoris cum numero terminorum continue creſcentis, quem ſumma progreſſionis illius geometricæ obtinet.

Idem valet de ſummis nonnullarum aliarum progreſſionum, quales ſunt ſeries numerorum figuratorum reciprocorum, aut æquimultiplorum ſive ſubmultiplorum eorundem.

Sit v. gr. $S = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} \ldots\ldots + \frac{1}{n-1.n} + \frac{1}{n.n+1}$.

$$\begin{aligned}
\text{Erit } S = {} & \tfrac{1}{1} - \tfrac{1}{2} \\
& + \tfrac{1}{2} - \tfrac{1}{3} \\
& + \tfrac{1}{3} - \tfrac{1}{4} \\
& + \tfrac{1}{4} - \tfrac{1}{5} \\
& + \tfrac{1}{5} - \tfrac{1}{6} \\
& + \tfrac{1}{6} \ldots\ldots\ldots \\
& \quad \vdots \\
& - \tfrac{1}{n-1} \\
& + \tfrac{1}{n-1} - \tfrac{1}{n} \\
& + \tfrac{1}{n} - \tfrac{1}{n+1}. \\
= {} & 1 \ldots\ldots\ldots\ldots\ldots - \tfrac{1}{n+1}.
\end{aligned}$$

Auctô n; quantitas $\frac{1}{n+1}$ poteſt reddi minor quacunque quantitate propoſita. Ideo 1 limes eſt ſummæ creſcentis progreſſionis illius.

Diſtinctio inter valorem & limitem quantitatis per incrementa continuè minora mutabilis apte illuſtratur exemplo ſpatiorum aſymptoticorum; quorum magnitudo ſæpe limitatur, quamvis aliqua dimenſionum eorundem ſine fine creſcere poſſit. Sic ſpatium inter duas lineas ordinatim applicatas curvæ loga-

logarithmicæ (ad Aſymptotum tanquam Axem relatæ) comprehenſum, accurate æquale eſt rectangulo facto ex ſubtangente & differentia illarum applicatarum. Atqui minor harum ordinatarum poteſt reddi minor lineâ quacunque magnitudine datâ; proinde major harum ordinatarum limes eſt differentiæ creſcentis earundem. Et rectangulum factum ex ſubtangente & majori ordinata, eſt limes ſpatii logarithmici creſcentis, quem ſpatium hoc nunquam attingit, ſed ad quem accedere poteſt propius quam ſpatium quodcunque dato prædicto rectangulo minus.

Exemplum 2dum. Demonſtrante ARCHIMEDE (*de Sphæra & Cylindro*) licet circulo inſcribere & circumſcribere polygona ordinata cognomina ſeu ſimilia, quorum ambitus & ſuperficies ab ambitu & ſuperficie circuli differunt minus quam ullâ lineâ aut ullâ ſuperficie aſſignatâ. Jam vero ambitus & ſuperficies circuli majores ſunt perimetris & ſuperficiebus polygonorum inſcriptorum; minores vero perimetris & ſuperficiebus polygonorum circumſcriptorum. Ideo ambitus & ſuperficies circuli ſunt reſpectivè limites perimetrorum & ſuperficierum (creſcentium aut decreſcentium) polygonorum ordinatorum, quæ circulo inſcribuntur, aut eidem circumſcribuntur. Eodem modo ſe habent ſuperficies & capacitates cylindrorum, conorum, ſphærarum, ad ſuperficies ac ſolida priſmatum, pyramidum, polyhedrorum, illis inſcriptorum aut circumſcriptorum.

Item: Ratio æqualitatis limes eſt rationum creſcentium aut decreſcentium, quas perimetri aut ſuperficies polygonorum regularium circulo inſcriptorum aut circumſcriptorum habent ad ambitum & ſuperficiem circuli. Idem valet de rationibus, quas cylindri, coni, ſphæræ habent ad priſmata, pyramides, polyhedra, ipſis inſcripta aut circumſcripta.

Exemplum 3tium. Sit curva quæcunque ad axem aliquem relata. Rectæ huic axi ordinatim applicatæ continuo creſcant; a zero usque ad maximam ipſarum, quæ ſumatur pro baſi. Axis dividatur in partes quotcunque inter ſe æquales. Per omnia puncta diviſionis ducantur ad curvam rectæ axi ordinatim applicatæ. Super baſi & omnibus rectis ordinatim applicatis conſtruantur verſus verticem parallelogramma, quorum altera latera ſint partes æquales

les axis. Hæc parallelogramma dicantur curvæ circumſcripta. Pariter ſuper omnibus rectis ordinatim applicatis baſi parallelis conſtituantur parallelogramma ad partes baſis, quorum altera latera ſint partes æquales axis. Hæc parallelogramma dicantur curvæ inſcripta. (Vide Fig. 1am.)

Exceſſus quô ſumma parallogrammorum circumſcriptorum ſuperat ſummam parallelogrammorum inſcriptorum; æqualis eſt parallelogrammo circumſcripto omnium maximo. Atqui baſis hujus parallelogrammi datur magnitudine: ergo, imminutâ altitudine, parallelogrammum hoc poteſt fieri minus quocunque ſpatio dato. Ideo differentia prædictarum ſummarum poteſt fieri minor quocunque ſpatio dato. Jam vero ſuperficies curvæ major eſt ſummâ parallelogrammorum inſcriptorum, ſed eadem minor eſt ſummâ parallelogrammorum circumſcriptorum; & quantitas, quâ ab alterutra harum ſummarum differt, minor eſt differentiâ mutuâ prædictarum ſummarum. Ergo figura curvilinea limes eſt, tam ſummæ creſcentis parallelogrammorum ipſi inſcriptorum, quam ſummæ decreſcentis parallelogrammorum eidem circumſcriptorum.

Rectæ axi ordinatim applicatæ ponantur eidem perpendiculares. Solidum rotatione prædictæ figuræ circa axem genitum limes erit tam ſummæ decreſcentis cylindrorum ſimul genitorum rotatione rectangulorum curvæ circumſcriptorum, quam ſummæ creſcentis cylindrorum ſimul genitorum rotatione rectangulorum curvæ inſcriptorum.

§. 2. *Scholium.* Sit quantitas aliqua data limes quantitatis mutabilis creſcentis vel decreſcentis. Utrique illarum addatur, aut ab utraque ſubtrahatur eadem aliqua quantitas. Prior ſumma, aut prior differentia, limes etiam erit poſterioris ſummæ, aut differentiæ, creſcentis vel decreſcentis. Si vero utraque quantitas ab aliqua eadem quantitate ſubtrahatur: prius reſiduum limes erit poſterioris decreſcentis aut creſcentis.

Fig. 2. Nempe denotet AB quantitatem datam, quæ limes ſit quantitatis mutabilis v. gr. creſcentis AX. Utrique illarum addatur, aut ab utraque ſubtrahatur eadem quantitas AC aut AC': prior ſumma CB, aut prior differentia $C'B$, limes quoque eſt creſcentis poſterioris ſummæ CX, aut poſterioris diffe-

feren-

ferentiæ $C'X$. Si vero utraque illarum ſubtrahatur ab eadem quantitate AC'': prius reſiduum $C''B$ limes eſt poſterioris reſidui decreſcentis $C''X$. Quod per ſe patet.

§. 3. *Theorema.* Sit quantitas data limes quantitatis mutabilis creſcentis vel decreſcentis. Dico rationem æqualitatis limitem eſſe rationis decreſcentis vel creſcentis, prioris quantitatis ad poſteriorem.

Nempe. Sit quantitas data AB limes quantitatis creſcentis v. gr. AX. Dico rationem $AB : AX$ poſſe accedere ad rationem æqualitatis, propius quam ad eam accedit data quæcunque ratio majoris ad minorem.

Demonſtratio. Quæcunque detur ratio majoris ad minorem; fiat ipſi æqualis ratio AB ad AD, quæ minor erit quam AB. Et fiat $AX > AD$ (quod poſſibile eſt per hyp.) Erit $AB : AX < AB : AD$. (a)

Eodem modo demonſtratur de altero Limite.

Viciſſim ſi ratio æqualitatis fuerit limes rationis decreſcentis aut creſcentis alicujus quantitatis datæ ad quantitatem mutabilem: Dico quantitatem datam limitem eſſe quantitatis mutabilis creſcentis aut decreſcentis.

Sit (ex. gr.) ratio æqualitatis limes rationis decreſcentis quantitatis datæ AB ad quantitatem mutabilem AX. Dico: quantitatem mutabilem AX poſſe fieri majorem quacunque quantitate datâ AD, quæ minor eſt quam AB.

Etenim per hyp. ratio $AB : AX$ poteſt fieri minor ratione data $AB : AD$ majoris ad minorem. Factum ſit: et cum $AB : AX < AB : AD$
erit $AX > AD$.

Eodem modo demonſtratur de altero limite.

Exempla. Cum perimeter & ſuperficies circuli limites ſint perimetrorum & ſuperficierum creſcentium aut decreſcentium polygonorum ordinatorum, quæ circulo inſcribuntur & circumſcribuntur: ratio æqualitatis eſt limes rationum decre-

(a) Quoniam demonſtrationes plerarumque propoſitionum hoc Capite evolutarum inæqualium rationum proprietatibus nituntur: ſi cui non fuerint ſatis notæ; ille adeat eximiam Diſſertationem inſcriptam: *Propoſitionum de Rationibus inter ſe diverſis Demonſtrationes, ex ſolis Libri V. Element. definitionibus & propoſitionibus deductæ;* quam Præſide celeb. Prof. PFLEIDERER publice propoſuit & defendit ornat. Candid. HAUBER. *Tubing.* 1793.

decreſcentium aut creſcentium perimetri & ſuperficiei circuli, ad perimetros & ſuperficies polygonorum regularium ipſi inſcriptorum aut circumſcriptorum. Idem valet de ſuperficiebus & capacitatibus cylindrorum, conorum, ſphærarum, reſpectu priſmatum, pyramidum, polyhedrorum, quæ ipſis inſcribuntur & circumſcribuntur.

Item: cum area figuræ §. 1. Ex. 3. Fig. 1. limes ſit ſummarum creſcentium aut decreſcentium parallelogrammorum, quæ ipſi inſcribuntur aut circumſcribuntur: ratio æqualitatis eſt limes rationis decreſcentis aut creſcentis ejusdem figuræ ad ſummas prædictas. Idemque valet de ſolidis ibidem commemoratis; & de ſummis cylindrorum, qui ipſis inſcribuntur & circumſcribuntur.

§. 4. *Theorema.* Sint duæ quantitates homogeneæ limitum capaces; ad quos vel utraque quantitas creſcendo, vel utraque quantitas decreſcendo, continue propius accedit. Sitque ratio harum quantitatum mutabilium ſemper eadem; ſeu æqualis rationi datæ: dico rationem limitum ipſarum æqualem eſſe eidem rationi datæ.

Fig. 3. Sint AB, CD, limites duarum quantitatum mutabilium AX, CY. Sit ratio $AX : CY$ ſemper æqualis rationi datæ $m : n$: dico rationem limitum AB, CD æqualem eſſe eidem rationi datæ $m : n$.

1°. 1°. Sint AB, CD limites quantitatum mutabilium creſcentium AX, CY.

Si ratio $AB : CD$ non ſit æqualis rationi datæ $m : n$; erit ea major aut minor. Tum vero, priore quidem caſu, ratio $m : n$ æqualis erit rationi alicujus quantitatis minoris ipſâ AB ad CD; altero autem caſu, ratio $m : n$ æqualis erit rationi AB ad aliquam quantitatem minorem quam CD. Proinde utroque caſu unus ex limitibus minui deberet, ut ratio limitis ita imminuti ad alterum limitem fieret æqualis rationi $m : n$.

Sit itaque, ſi fieri poſſit: $m : n = AB : CD' \; (< CD)$. Sumatur $CY > CD'$ (quod fieri poteſt per hyp.)

Tum fiat $n : m = CY : AX$
Atqui $n : m = CD' : AB$
Ergo $CY : AX = CD' : AB$
Sed $CY > CD'$ (*conſtr.*)
Ergo $AX > AB$ contra hypotheſin; cum AB ſit limes quantitatis creſcentis AX.

2°.

2°. Sint AB, CD limites quantitatum mutabilium decreſcentium AX, CY. Fig. 3. 2°.

Si ratio $AB : CD$ non ſit æqualis rationi datæ $m : n$; rurſus erit ea major aut minor. Tum priore caſu ratio $m : n$ æqualis erit rationi ipſius AB ad aliquam quantitatem majorem ipſâ CD; altero caſu ratio $m : n$ æqualis erit rationi alicujus quantitatis majoris ipſâ AB ad CD. Ideo utroque caſu unus ex limitibus augeri deberet, ut ratio hujus limitis ita aucti ad alterum limitem fieret æqualis rationi $m : n$.

Sit itaque $m : n = AB : CD' \,(> CD)$. Sumatur $CY < CD'$ (quod fieri poteſt per hyp.): Et ſit $n : m = CY : AX$

Atqui $n : m = CD' : AB$

Ergo $CY : AX = CD' : AB$

Sed $CY < CD'$

Ergo $AX < AB$, contra hypotheſin; cum AB ſit limes quantitatis decreſcentis AX.

Ratio igitur $m : n$ neque major eſt neque minor ratione limitum. Proinde ratio limitum æqualis eſt rationi $m : n$.

Obſervatio. Hæc propoſitio unum eſt ex præcipuis fundamentis methodi exhauſtionis, qualis ſedulo fuit ab Antiquis exculta, atque ab Euclide & Archimede nobis transmiſſa.

En aliquot exempla illam illuſtrantia.

Perimetri polygonorum ordinatorum ſimilium, circulis diverſis inſcriptorum, ſunt in ratione data radiorum circulorum, intra vel circa quos deſcribuntur. Sed perimetri circulorum ſunt limites perimetrorum creſcentium polygonorum ipſis inſcriptorum, & limites perimetrorum decreſcentium polygonorum circumſcriptorum. Itaque perimetri circulorum ſunt inter ſe in eadem ratione data; ſcilicet in ratione horum radiorum.

Item: ſuperficies circuli limes eſt ſuperficierum creſcentium aut decreſcentium polygonorum ordinatorum, quæ ipſi inſcribuntur aut circumſcribuntur. Sed ſuperficies polygonorum ſimilium diverſis circulis inſcriptorum aut circumſcriptorum ſunt in ratione duplicata radiorum prædictorum circulorum. Proinde ſuperficies circulorum ſunt etiam in ratione duplicata radiorum ſuorum.

Sint duæ pyramides æque altæ, quarum baſes in eodem plano, & quæ ſitæ ſint ad easdem partes hujus plani. Altitudo communis harum pyramidum ſecetur in quotcunque partes æquales: per puncta diviſionis agantur plana baſi parallela; & utrique pyramidi inſcribantur & circumſcribantur prismata, ad normam eorum, quæ de parallelogrammis inſcriptis & circumſcriptis §. 1. Ex. 3. dicta fuerunt. Utraque pyramis limes eſt ſummarum creſcentium aut decreſcentium prismatum, quæ ipſi inſcribuntur & circumſcribuntur. Proinde demonſtrato, prædictas prismatum correſpondentium ſummas inter ſe eſſe in ratione data baſium pyramidum; licebit concludere: rationem limitum harum ſummarum, ſeu ipſarum pyramidum, æqualem eſſe eidem rationi datæ baſium.

Eodem modo demonſtratur: tam cylindros quam conos æquè-altos eſſe inter ſe uti baſes: cum ſolida hæc limites ſint prismatum aut pyramidum, quæ ipſis inſcribuntur & circumſcribuntur.

Hinc etiam ſi cylinder & conus æquè alti eidem baſi inſiſtant: cylinder eſt triplus coni.

Hinc rurſus ſuperficies tam cylindrorum quàm conorum ſimilium ſunt in duplicata ratione dimenſionum ſuarum homologarum; ſoliditates autem in eadem ratione triplicata.

Hinc tandem fluunt nunquam ſatis laudata Archimedis inventa de ſuperficie & ſoliditate ſphærarum.

Super diametro circuli tanquam axe deſcribatur ellipſis quæcunque. Diameter hæc ſecetur in partes quotcunque æquales; & utrique figuræ inſcribantur & circumſcribantur rectangula ad normam eorum, quæ §. 1. Ex. 3. constructa fuerunt. Circulus & ellipſis limites reſpectivè ſunt tam ſummarum creſcentium parallelogrammorum, quæ ipſis inſcribuntur, quam ſummarum decreſcentium parallelogrammorum, quæ ipſis circumſcribuntur. Sed ſummæ rectangulorum circulo & ellipſi inſcriptorum aut circumſcriptorum ſunt inter ſe in ratione conſtanti prioris axis ellipſis ad axem ei conjugatum. Ergo etiam circulus & ellipſis ſunt inter ſe in ratione data horum axium.

Similiter comparatio inſtituitur ſphæræ & ellipſoïdis, rotatione circuli & ellipſis

ellipſis circa eundem axem genitarum. Nempe ſolida hæc reſpectivè limites ſunt ſummarum cylindrorum, qui ipſis inſcribuntur & circumſcribuntur. Atqui ſummæ iſtæ ſunt inter ſe in ratione conſtanti, quæ eſt duplicata rationis hujus axis ad axem ipſi conjugatum: ergo etiam ſphæra & ellipſois ſunt inter ſe in eadem ratione duplicata.

Utilitas præſentis propoſitionis plurimis aliis applicationibus illuſtrari poſſet. Sed hæc abunde ſufficiant.

§. 5. *Theorema.* Duæ quantitates mutabiles heterogeneæ, ſive creſcentes ambæ, ſive ambæ decreſcentes, ſint limitum capaces: et rationes harum quantitatum ad duas quantitates datas ſint ſemper inter ſe æquales. Dico: rationes limitum harum quantitatum mutabilium, ad eaſdem quantitates datas, eſſe etiam inter ſe æquales.

Sint Q & Q' duæ quantitates mutabiles. Sint L & L' limites harum quantitatum ſimul creſcentium, aut ſimul decreſcentium. Sint C & C' duæ quantitates conſtantes. Sit ſemper $Q : C = Q' : C'$
Dico eſſe $L : C = L' : C'$.

Primus Caſus. Uterque limes L & L' ſit limes quantitatum Q & Q' creſcentium.

Si non eſt $L : C = L' : C'$; alterutra rationum $L : C$, $L' : C'$; major erit altera; & proinde minuendus erit antecedens majoris rationis, ut alteri fiat æqualis. Sit igitur, ſi fieri poſſit: $L - q : C = L' : C'$.

Fiat $Q > L - q$ (quod fieri poteſt per hyp.)
Erit $Q : C > L - q : C$
Sed $Q : C = Q' : C'$ (hyp.)
Et $L - q : C = L' : C'$.
Ergo $Q' : C' > L' : C'$
ideo $Q' > L'$ (contra hyp.)

Secundus Caſus. Uterque limes L & L' ſit limes quantitatum Q & Q' decreſcentium.

Si non eſt $L : C = L' : C'$: alterutra rationum $L : C$, $L' : C'$; minor erit alterâ; & proinde augendus erit antecedens minoris rationis, ut alteri fiat æqualis. Sit ideo, ſi fieri poſſit: $L + q : C = L' : C'$.

 Fiat

$$\begin{array}{llcll}
\text{Fiat} & Q & < & L+q & \text{(quod fieri poteſt per hyp.)} \\
\text{Erit} & Q : C & < & L+q : C & \\
\text{Sed} & Q : C & = & Q' : C' & \\
\text{Et} & L+q : C & = & L' : C' & \\
\text{Ergo} & Q' : C' & < & L' : C' & \\
\text{et ideo} & Q' & < & L' & \text{(contra hyp.)}
\end{array}$$

Neutra ideo rationum $L : C$, $L' : C'$, major aut minor eſt alterâ. Proinde hæ duæ rationes ſunt inter ſe æquales; ſeu $L : C = L' : C'$.

Obſervatio. Hæc propoſitio pariter eſt unum ex fundamentis methodi exhauſtionis Veterum. Illius ope menſuram obtinere licet plurimorum extenſorum, datâ menſura aliarum quarundam magnitudinum; uti paucis oſtendam exemplis, deſumptis a quadratura parabolæ conicæ, & cubatura paraboloïdis rotatione parabolæ hujus circa axem genitæ.

Fig. 4. Sit nempe *SMAB* dimidium ſegmentum parabolæ conicæ, cujus vertex *S*, abſciſſa axis *SB*, & baſis axi ordinatim applicata *AB*. Et quæratur area hujus ſpatii.

Per verticem *S*, ductâ tangente *SD*, compleatur rectangulum *ABSD*, & jungatur *SA* recta. Dividatur tangens *SD* in partes quotcunque inter ſe æquales, quarum una ſit PP'. Spatio parabolico exteriori *SDAM*, & triangulo *SDA* circumſcribantur (aut inſcribantur) rectangula æquè-alta (ad normam eorum, quæ §. 1. Ex. 3. ſunt deſcripta). Sint $PMmP'$, $PQqP'$, duo rectangula ſibi mutuò reſpondentia, ſpatio illi & triangulo v. gr. circumſcripta. Ordinatæ *PM*, $P'M'$ occurrant baſi *AB* in punctis *R* & R'.

Concipiatur triangulum *SAD*, & rectangulum *SBAD*; gyrari circa tangentem *SD* tanquam axem. Triangulum *SAD* gignet conum, cui circumſcribentur cylindri rectangulis *Pq* geniti; & rectangulum *SBAD* gignet cylindrum.

$$\begin{array}{rcl}
\text{Per naturam parabolæ} \quad MP : AD & = & SP^2 : SD^2 \\
 & = & PQ^2 : AD^2 \\
 & = & PQ^2 : PR^2 \\
 & = & \text{cyl.}\,PQqP' : \text{cyl.}\,PRR'P' \\
\text{Sed} \quad MP : AD & = & MP : PR \\
 & = & \text{rect.}\,PMmP' : \text{rect.}\,PRR'P'. \\
\text{Ergo} \quad \text{rect.}\,PMmP' : \text{rect.}\,PRR'P' & = & \text{cyl.}\,PQqP' : \text{cyl.}\,PRR'P'.
\end{array}$$

Alti-

Altitudine PP' manente eâdem; termini conſequentes omnium ſimilium proportionum ſunt etiam conſtantes.

$$\text{Hinc} \quad \text{ſ. rect. } PMmP' : \text{ſ. rect. } PRR'P' = \text{ſ. cyl. } PQqP' : \text{ſ. cyl. } PRR'P'$$

$$\text{ſeu} \quad \text{ſ. rect. } PMmP' : ABSD = \text{ſ. cyl. } PQqP' : \text{cyl. } ABSD.$$

$$\text{Hinc (§. praeſ. 1) lim. ſ. rect. } PMmP' : ABSD = \text{lim. ſ. cyl. } PQqP' : \text{cyl. } ABSD.$$

$$\text{Sed (§. 1.)} \quad \text{lim. ſ. rect. } PMmP' = \text{ſpatio parabolico } AMSD$$

$$\text{et} \quad \text{lim. ſ. cyl. } PQqP' = \text{cono gyratione trianguli } SAD \text{ genito.}$$

$$\text{Ergo} \quad AMSD : ABSD = \text{con. } ASD : \text{cyl. } ABSD$$

$$\text{Atqui (Archim. de cono \& cyl.)} \quad \text{con. } ASD = \tfrac{1}{3} \text{ cyl. } ABSD$$

$$\text{Ergo} \quad AMSD = \tfrac{1}{3} ABSD$$

$$\text{Et} \quad AMSB = \tfrac{2}{3} ABSD.$$

Hoc eſt: dimidium ſegmentum parabolæ, abſciſſa axis, recta axi ordinatim applicata, & arcu curvæ contentum, æquale eſt duobus trientibus rectanguli circumſcripti.

Porro: ſit SAB dimidium ſegmentum parabolicum, quod gyratione ſua circa axem SB gignat ſegmentum paraboloïdicum. Fig. 5.

Jungatur SA recta. Ductâ per verticem S, tangente SD, compleatur rectangulum $ABSD$. Dividatur axis SB in partes quotcunque æquales, quarum una ſit PP'. Spatio parabolico & triangulo SAB circumſcribantur (vel inſcribantur) rectangula æquè alta (ad normam §. 1. Ex. 3.). Sint $PMmP'$, $PQqP'$ duo rectangula ſibi invicem reſpondentia, ſegmento parabolico & triangulo circumſcripta, & rectæ PM, $P'M'$, axi ordinatim applicatæ, occurrant rectæ AD in punctis R & R'.

$$\text{Per naturam parabolæ eſt} \quad SP : SB = PM^2 : AB^2$$
$$= PM^2 : PR^2$$
$$= \text{cyl. } PMmP' : \text{cyl. } PRR'P'$$

$$\text{Atqui} \quad SP : SB = PQ : AB$$
$$= PQ : PR$$
$$= \text{rect. } PQqP' : \text{rect. } PRR'P'$$

$$\text{Ergo} \quad \text{rect. } PQqP' : \text{rect. } PRR'P' = \text{cyl. } PMmP' : \text{cyl. } PRR'P'.$$

Altitudine PP' manente eadem: termini conſequentes omnium ſimilium proportionum ſunt etiam conſtantes.

Ergo

Ergo ſ. rect. $PQqP'$: ſ. rect. $PRR'P'$ $=$ ſ. cyl. $PMmP'$: ſ. cyl. $PRR'P'$

ſeu ſ. rect. $PQqP'$: rect. $ABSD$ $=$ ſ. cyl. $PMmP'$: cyl. $ABSD$

Ergo lim. ſ. rect. $PQqP'$: rect. $ABSD$ $=$ lim. ſ. cyl. $PMmP'$: cyl. $ABSD$

Sed (§. 1. Ex. 3.) lim. ſ. rect. $PQqP'$ $=$ triang. ASB

lim. ſ. cyl. $PMmP'$ $=$ paraboloïdi $AMSB$.

Ergo triang. ASB : rect. $ABSD$ $=$ parab. $AMSB$: cyl. $ABSD$

Sed triang. $ASB = \frac{1}{2}$ rect. $ABSD$

Ergo - - - - - - - - parab. $AMSB = \frac{1}{2}$ cyl. $ABSD$.

Hoc eſt: paraboloïdes gyratione dimidii ſegmenti parabolici circa axem ſuum genita eſt ſubdupla cylindri circumſcripti.

Scholium. Exempla hæc declarant, quomodo cognita ratione duorum extenſorum ſæpius determinari poſſit ratio, quam invicem habent duo alia extenſa prioribus heterogenea. In priori exemplo, ratio data, quam mutuò habent conus & cylindrus æquè alti ſuper eadem baſi, deduxit ad rationem, quæ intercedit inter dimidium ſegmentum parabolæ & rectangulum circumſcriptum; in poſteriori, ratio data trianguli & rectanguli æquè altorum ſuper eadem baſi idem præſtitit pro paraboloïde & cylindro huic circumſcripto.

§. 6. *Theorema.* Sint duæ aut plures quantitates mutabiles, quæ ſimul fieri poſſint minores quacunque quantitate propoſita: dico earum ſummam poſſe etiam quacunque data quantitate fieri minorem.

Sint $x, y, z, v \ldots\ldots$ quantitates mutabiles, quarum numerus n, quæ ſimul fieri poſſint quacunque quantitate propoſita minores: dico, ſummam harum quantitatum mutabilium poſſe fieri quacunque quantitate data a minorem.

Etenim dividatur quantitas a in tot partes æquales, quot ſunt quantitates mutabiles. Fiant ſimul (quod poſſibile per hyp.) ſingulæ quantitates mutabiles unâ illarum partium minores. Proinde ſumma omnium quantitatum mutabilium minor erit prædicta parte, toties ſumptâ quot ſunt quantitates mutabiles, ſeu minor quantitate propoſita.

Corollarium 1. Idem a fortiori valet de differentia duarum ejusmodi quantitatum mutabilium, & de exceſſu, quo ſumma aliquot ejusmodi quantitatum mutabilium ſuperat ſummam reliquarum.

Corol-

Corollarium 2. Sit quantitas mutabilis, quæ minor fieri poteſt quacunque quantitate data: & ea quantitas multiplicetur per numerum quemcunque n (poſitivum). Dico, productum inde ortum poſſe etiam minus fieri quacunque quantitate propoſita.

Sit n numerus integer: res patet per Propoſitionem.

Si vero n non ſit numerus integer: fiat productum ex quantitate mutabili per numerum integrum, ipſo n proxime majorem, minus quantitate data: erit a fortiori productum ex quantitate mutabili per numerum n minus eadem quantitate data.

§. 7. *Theoremata varia.*

1°. Si rationis mutabilis creſcentis v. gr. (*a*) $X : Y$ limes ſit ratio data $A : B$. Erit (invertendo) rationis decreſcentis $Y : X$ limes ratio $B : A$.

Quoniam (per hyp.) $X : Y \lessdot A : B$

eſt $Y : X \gtrdot B : A$. Sit $B' \gtrdot B$, & proinde $B' : A \gtrdot B : A$;

dico fieri poſſe $Y : X \lessdot B' : A$.

Etenim per hyp. fieri poteſt $X : Y \gtrdot A : B'$,

fiat; & erit $Y : X \lessdot B' : A$.

2°. Si rationis mutabilis creſcentis v. gr. $X : Y$ limes ſit ratio data $A : B$: erit (convertendo) rationis mutabilis decreſcentis $X : X - Y$, limes ratio data $A : A - B$.

Quoniam (per hyp.) $X : Y \lessdot A : B$

eſt $X : X - Y \gtrdot A : A - B$. Sit autem $B' \gtrdot B$, & proinde $A - B' \lessdot A - B$ & $A : A - B' \gtrdot A : A - B$

Dico fieri poſſe $X : X - Y \lessdot A : A - B'$.

Etenim quoniam $B' \gtrdot B$; eſt $A : B' \lessdot A : B$;

Sed (per hyp.) fieri poteſt $X : Y \gtrdot A : B'$,

Fiat; & erit - - - $X : X - Y \lessdot A : A - B'$.

3°. Si rationis mutabilis creſcentis v. gr. $X : Y$, limes ſit ratio data $A : B$; erit componendo / dividendo rationis mutabilis creſcentis $X \pm Y : Y$ limes, ratio data $A \pm B : B$.

(*a*) Quæcunque dicam de ratione creſcente, dicta intelligantur de ratione decreſcente, mutatis mutandis.

Quoniam (per hyp.) $X : Y < A : B$

$X \pm Y : Y < A \pm B : B$: Sit autem $A' < A$; & proinde
$A' \pm B < A \pm B$
& $A' \pm B : B < A \pm B : B$

Dico fieri poſſe $X \pm Y : Y > A' \pm B : B$

Etenim quoniam $A' < A$

$A' : B < A : B$

Sed (per hyp.) fieri poteſt $X : Y > A' : B$

Fiat; & erit $X \pm Y : Y > A' \pm B : B$.

§. 8. *Theorema.* Duæ quantitates mutabiles ſemper inter ſe ſint in ratione data. Et quantitas data ſit limes unius harum quantitatum mutabilium creſcentis vel decreſcentis: dico, dari etiam quantitatem, quæ ſit limes alterius quantitatis, creſcentis vel decreſcentis, & quantitatem hanc eam eſſe, ad quam prior data quantitas habet eandem rationem datam.

Fig. 3. 1°. Sint AX, CY duæ quantitates mutabiles datam inter ſe habentes rationem $a : c$. Sit AB quantitas data, quæ ſit limes quantitatis mutabilis AX creſcentis v. gr: dico, dari alteram quantitatem CD, quæ erit limes alterius quantitatis mutabilis creſcentis CY; & quantitatem hanc CD eam eſſe, ad quam quantitas data AB habet rationem datam $a : c$.

Etenim fiat $a : c = AB : CD$

per hyp. $AX : CY = a : c$

Ergo $AX : CY = AB : CD$; jam vero $AB > AX$ (hyp.); ergo $CD > CY$

hinc $AB - AX : CD - CY = AB : CD$

ſeu $XB : DY = AB : CD = a : c$; & $DY = BX \times \frac{c}{a}$.

Sed (per hyp.) BX poteſt fieri minor qualibet quantitate propoſita; ergo (§. 6.) & DY ſimul minor reddi poteſt quacunque quantitate propoſita. Et proinde quantitas CD, ſemper major mutabili CY, limes eſt magnitudinis hujus quantitatis CY.

Eodemque modo propoſitum oſtenditur de limite decrementi.

§. 9. *Theorema.* Duæ quantitates mutabiles eandem ſemper habeant rationem ad duas quantitates datas A & A'. Detur quantitas C, quæ limes ſit

unius

unius illarum quantitatum ſive creſcentis ſive decreſcentis. Dico, dari etiam aliquam quantitatem C', quæ ſit limes poſterioris quantitatis mutabilis pariter creſcentis aut decreſcentis; & quantitatem hanc C' eam eſſe, quâ fiat $C : A = C' : A'$. Demonſtratur eodem modo quo Theorema præcedens.

§. 10. *Theorema.* Duæ rationes mutabiles ſint ſemper inter ſe æquales; & ſit ratio aliqua data limes unius harum rationum mutabilium creſcentis vel decreſcentis: dico, eandem rationem datam eſſe etiam limitem poſterioris rationis creſcentis vel decreſcentis.

Sit ſemper $X : X' = Y : Y'$.

Et ſemper ſit (v. gr.) $X : X' < A : C$; ſed lim. $X : X' = A : C$

Dico etiam - - - - - - lim. $Y : Y' = A : C$.

Etenim (per hyp.) $X : X' = Y : Y'$

Sed $X : X' < A : C$

Ergo $Y : Y' < A : C$. Sit $C' > C$; & proinde $A : C > A : C'$.

Fieri poteſt (per hyp.) $X : X' > A : C'$;

Fiat; & erit $Y : Y' > A : C'$. Proinde ratio $A : C$ etiam eſt limes incrementi rationis $Y : Y'$.

Corollarium. Si ratio aliqua data limes eſt rationis duarum quantitatum mutabilium: eadem ratio data limes quoque eſt rationis quantitatum, quæ ſunt æquè-multiplæ vel æquè-ſubmultiplæ poſteriorum.

§. 11. *Theorema.* Sint duæ quantitates datæ, quarum una ſit limes alicujus quantitatis mutabilis decreſcentis; altera vero limes ſit alterius quantitatis mutabilis creſcentis; ita ut magnitudines, quibus duæ quantitates mutabiles a limitibus ſuis reſpectivè differunt, ſimul fieri poſſint minores quibuscunque quantitatibus propoſitis. Dico: rationem, quam prior quantitas data habet ad poſteriorem, eſſe limitem rationis decreſcentis, quam prior quantitas mutabilis habet ad poſteriorem.

Sit AB limes quantitatis mutabilis decreſcentis AX; & CD limes quantitatis mutabilis creſcentis CY. Quoniam $\begin{matrix} AX > AB \\ CY < CD \end{matrix}$, eſt $AX : CY > AB : CD$. Fig. 6.
Dico: rationem $AX : CY$ poſſe reddi minorem quacunque ratione propoſita, quæ major ſit ratione $AB : CD$.

Sit

Sit enim propoſita ratio quæcunque $AB : CE$ major ratione $AB : CD$. Erit $CE < CD$. Porro, ſit quantitas $CF \genfrac{}{}{0pt}{}{> CE}{< CD}$; & fiat, uti $AB : CE$, ita $AG : CF$. Quoniam $CF > CE$, erit $AG > AB$. Fiat ſimul $\genfrac{}{}{0pt}{}{AX < AG}{CY > AF}$ (quod fieri poteſt per hyp.)

Dico, fore $AX : CY < AB : CE$

Quoniam $AX < AG$; . . . $AX : CF < AG : CF$.

$< AB : CE$

Atqui $CF < CY$; ergo $AX : CY < AX : CF$

Ergo a fortiori . . . . $AX : CY < AB : CE$.

Obſervatio. Hinc ratio poſterioris quantitatis datæ ad priorem eſt limes rationis creſcentis poſterioris quantitatis mutabilis ad priorem (§. 1.)

§. 12. *Theorema.* Sint duæ quantitates datæ, quæ ambæ limites ſunt duarum quantitatum mutabilium ſimul decreſcentium; ita ut exceſſus, quibus limites ſuos ſuperant, poſſint ſimul fieri minores quibuslibet magnitudinibus datis. Dico, rationem duarum quantitatum mutabilium ſimul fieri poſſe majorem quacunque ratione datâ, quæ ſit minor ratione prioris limitis ad poſteriorem, minorem vero quacunque ratione datâ, quæ major ſit ratione prioris limitis ad poſteriorem.

Fig. 7. Sint AB, CD limites duarum quantitatum mutabilium decreſcentium AX, CY. Dico, rationem $AX : CY$ poſſe fieri ſimul majorem quacunque ratione datâ $AB : CE$, quæ ſit minor ratione $AB : CD$; & minorem ratione datâ $AF : CD$, quæ ſit major ratione $AB : CD$.

Demonſtratio. Etenim, fiant ſimul $\genfrac{}{}{0pt}{}{AX < AF}{CY < CE}$ (quod fieri poteſt per hyp.)

1°. Quoniam $AX > AB$, eſt $AX : CE > AB : CE$

Atqui $CY < CE$; ergo $AX : CY > AX : CE$

Ergo a fortiori . . . . . $AX : CY > AB : CE$

2°. Quoniam $CY > CD$, eſt $AF : CY < AF : CD$

Atqui $AX < AF$; ergo $AX : CY < AF : CY$

Ergo a fortiori . . . . . $AX : CY < AF : CD$.

NB. Idem obtinebit, ſi quantitates datæ fuerint limites duarum quantitatum mutabilium creſcentium.

Obſer.

Obſervatio. Quamvis hisce caſibus ratio limitum AB & CD non poſſit dici ſemper eſſe major aut ſemper minor ratione quantitatum mutabilium AX, CY, quod ſtricto ſenſu juxta definitiones (§. 1.) requiritur, ut ratio data $AB:CD$ dici poſſit limes rationis mutabilis $AX:CY$; attamen, quoniam poſterior ratio, ſive ſit major priore, ſive eâ minor, ad illam ita accedere poteſt, ut minus ab ea differat, quam duæ rationes propoſitæ quæcunque, quarum una major altera minor eſt ratione limitum, ita ut nullus ſit exiguitatis quantitatum BF, DE limes, quo rationum $AB:CE$, $AF:CD$ ad rationem $AB:CD$ acceſſus cohiberetur; liquet tanto magis, rationis $AX:CY$ a ratione $AB:CD$ diſcrepantiam intra limites continue arctiores poſſe contineri. Et proinde ratio data $AB:CD$ dici quoque poteſt ea, ad quam ratio $AX:CY$ propius propiusque accedit, quamvis fluxu non æquè continuo & regulari, uti in omnibus exemplis antea propoſitis: cum ratio mutabilis $AX:CY$ alternis vicibus poſſit major aut minor fieri ratione data $AB:CD$.

Exemplo hoc ducimur ad alteram ſpeciem limitum rationum, paululum ab ea, quam §. 1. definivimus, diverſam.

§. 13. *Definitio.* Ratio mutabilis poſſit alternis vicibus fieri major aut minor ratione datâ; ita tamen ut ad rationem datam propius accedere poſſit, quam ad eandem accedit quæcunque alia ratio propoſita, ſive major priori ratione datâ, ſive minor: prior ratio data dicitur etiam limes rationis mutabilis; & heic quidem obtinet limes rationis tam creſcentis quam decreſcentis.

Idem diſcrimen locum etiam habere poteſt in limitibus quantitatum, quod ſequenti exemplo prorſus familiari declaratur.

Sit p minor unitate, & ſit progreſſio $1-p+p^2-p^3+p^4-p^5 \ldots \mp p^{n-2} \pm p^{n-1}$. Vera ſumma hujus progreſſionis eſt $\frac{1}{1+p} \pm \frac{p^n}{1+p}$; prouti n eſt numerus $\begin{matrix}\text{impar}\\ \text{par}\end{matrix}$. Proinde numero n alterne ſumto impari & pari, ſumma illa etiam alterne ſit major & minor quantitate $\frac{1}{1+p}$. Quare ſenſu ſtricto definitionis (§. 1.) quantitas $\frac{1}{1+p}$ dici nequit limes eſſe ſummæ ſeriei propoſitæ; cum hæc ſumma neque ſemper major ſit, neque ſemper minor quantitate $\frac{1}{1+p}$. Cum vero tam

exceſ-

exceſſus, quam defectus, quibus ſumma progreſſionis differt a quantitate $\frac{1}{1+p}$, poſſint quacunque quantitate propoſita fieri minores; quantitas $\frac{1}{1+p}$ jure etiam nunc dicitur limes ſummæ, ſive creſcentis, ſive decreſcentis, propoſitæ progreſſionis.

Idemque dicatur de plurimis progreſſionibus decreſcentibus, quæ alternè exceſſu & defectu differunt ab aliqua quantitate data; ita tamen ut tam exceſſus quam defectus poſſint fieri minores quacunque quantitate propoſita. Ex. gr. ſit p quadrans circumferentiæ, cujus radius $= 1$:

ſit $\frac{1}{2}p = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \ldots\ldots$

Et quamvis ſumma hujus ſeriei alternè ſit major & minor eadem quantitate data $\frac{1}{2}p$; hæc tamen absque dubio limes illius eſſe cenſetur. Sic & log. hyp. 2 dicitur limes ſummæ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\ldots$

Conjuncta utraque limitum notione, duo poſtrema theoremata brevius ſic enunciantur: *Ratio limitum duarum quantitatum mutabilium limes eſt rationis ipſarum quantitatum mutabilium.*

Exempla. Rationes tam perimetrorum quam ſuperficierum duorum circulorum limites ſunt rationum perimetrorum & ſuperficierum polygonorum regularium ſive ſimilium ſive diſſimilium, quæ circulis circumſcribuntur & inſcribuntur; ſive polygona circulis ſimul circumſcribantur, ſive ipſis ſimul inſcribantur; ſive tandem polygona ſint uni circulo circumſcripta, alteri inſcripta.

Pariter ſi absque limite imminui poſſunt quantitates x & y: ratio quantitatum a & b limes eſt rationum $a \pm x : b \pm y$.

§. 14. *Theorema.* Ratio compoſita ex rationibus, quæ limites ſunt duarum pluriumve rationum mutabilium, limes eſt rationis compoſitæ ex iisdem rationibus mutabilibus.

1°. Sint duæ rationes mutabiles ſemper reſpectivè $\substack{\text{majores}\\\text{minores}}$ duabus rationibus datis; ſed quæ fieri poſſint *ſimul* reſpectivè $\substack{\text{minores}\\\text{majores}}$ duabus rationibus quibuscunque $\substack{\text{majoribus}\\\text{minoribus}}$ quam prædictæ rationes datæ. Dico: rationem compoſitam ex duabus rationibus mutabilibus (quæ ſemper $\substack{\text{major}\\\text{minor}}$ eſt ratione compo-

ſita

ſita ex duabus rationibus datis), poſſe fieri $\begin{matrix}\text{minorem}\\ \text{majorem}\end{matrix}$ quacunque ratione data, quæ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ ſit ratione compoſita ex rationibus datis.

Cum modus demonſtrandi idem ſit pro utraque inæqualitatis ſpecie: ſufficiat (brevitatis cauſa) de altera illarum propoſitum evincere.

Sint igitur duæ rationes datæ $\begin{matrix} AB : BC \\ BC : CD \end{matrix}$, & ſint duæ rationes mutabiles $\begin{matrix} X:Y \\ Y:Z \end{matrix}$. 1°. Sit ſemper $\begin{matrix} X:Y > AB:BC \\ Y:Z > BC:CD \end{matrix}$; ſed duæ rationes $\begin{matrix} X:Y \\ Y:Z \end{matrix}$ ſimul fieri poſſint minores quibuscunque rationibus datis, quæ ſint rationibus $\begin{matrix} AB:BC \\ BC:CD \end{matrix}$ reſpectivè majores. Dico, rationem $X : Z$ (quæ componitur ex rationibus $\begin{matrix} X : Y \\ Y : Z \end{matrix}$) pariter minorem fieri poſſe quacunque ratione propoſita, quæ major eſt ratione $AB : CD$ (quæ componitur ex rationibus $\begin{matrix} AB : BC \\ BC : CD \end{matrix}$). Fig. 8.

Et primo quidem, quoniam $\begin{matrix} X : Y > AB : BC \\ Y : Z > BC : CD \end{matrix}$
erit ſemper $X : Z > AB : CD$.

Jam proponatur quælibet ratio $AB : CE$ major ratione $AB : CD$ (& proinde ſit $CE < CD$). Probandum eſt fieri poſſe $X : Y < AB : CE$.

Fiat, uti $BC : CD$, ita $BF : CE$. Et quoniam $CD > CE$; erit $BC > BF$. Fiat etiam (quod poſſibile per hyp.) $X : Y (< AB : BF) = AB : BG$ $(BG > BF)$; & fiat ſimul (quod poſſibile per hyp.) $Y : Z \left(\begin{matrix} > BC : CD \\ \text{ſeu } BF : CE \end{matrix}\right) < BG : CE$

Erit ideo $X : Z$ . . . . . . $< AB : CE$.

Nempe: ratio $X : Z$ potuit fieri minor ratione quacunque propoſita $AB : CE$, quæ major eſt ratione data $AB : CD$.

2°. Sint duæ rationes mutabiles, quarum una ſemper major ſit aliqua ratione data, altera vero ſemper ſit minor aliqua altera ratione data: ita tamen ut prior ratio mutabilis poſſit fieri minor quacunque ratione propoſita, qu major ſit priore ratione data; & *ſimul* poſterior ratio mutabilis poſſit fieri major quacunque ratione propoſita, quæ minor eſt poſteriori ratione data. Dico: rationem compoſitam ex duabus rationibus mutabilibus poſſe fieri tum majorem quacunque ratione propoſita, quæ minor fuerit ratione compoſita ex duabus prioribus rationibus datis; tum & minorem quacunque ratione propoſita, quæ major fuerit ratione compoſita ex prædictis rationibus datis.

Sint $\begin{matrix} X : \Upsilon \\ \Upsilon : Z \end{matrix}$ duæ rationes mutabiles, & ſint $\begin{matrix} AB : BC \\ BC : CD \end{matrix}$ duæ rationes datæ; ita ut ſemper $\begin{matrix} X : \Upsilon > AB : BC \\ \Upsilon : Z < BC : CD \end{matrix}$. Liquet nihil exinde concludi poſſe, quod ad inæqualitatem (imo & quod ad æqualitatem) rationum, quæ ex rationibus mutabilibus, & ex rationibus datis componuntur.

Sit autem ratio $AB : BC$ limes rationis decreſcentis $X : \Upsilon$; &
ſimul ſit ratio $BC : CD$ limes rationis creſcentis $\Upsilon : Z$.

Dico, rationem - - - - - - - - $X : Z$ poſſe fieri $\begin{matrix} \text{minorem} \\ \text{majorem} \end{matrix}$ quacunque ratione propoſita, quæ $\begin{matrix} \text{major} \\ \text{minor} \end{matrix}$ ſit ratione data $AB : CD$.

Fig. 9. 1°. Sit ratio propoſita $AE : CD$ major ratione $AB : CD$ (& proinde ſit $AE > AB$).

Fiat (quod poſſibile per hyp.)	$X : \Upsilon < AE : BC$
Sed eſt ſemper (per hyp.)	$\Upsilon : Z < BC : CD$
Ergo erit	$X : Z < AE : CD.$

2°. Sit ratio propoſita $AB : CF$, minor ratione $AB : CD$ (& proinde $CF > CD$).

Fiat (quod poſſibile per hyp.)	$\Upsilon : Z > BC : CF$
Quoniam eſt ſemper (per hyp.)	$X : \Upsilon > AB : BC$
& factum fuit	$\Upsilon : Z > BC : CF$
Erit	$X : Z > AB : CF.$

Proinde, propoſitis duabus rationibus, $AE : BC$, & $BC : CF$; quarum prior major ſit ratione data $AB : BC$, & poſterior minor ſit ratione data $BC : CD$; fiat ſimul $\begin{matrix} X : \Upsilon < AE : BC \\ \Upsilon : Z > BC : CF \end{matrix}$; erit ſimul $\begin{matrix} X : Z < AE : CD \\ X : Z > AB : CF \end{matrix}$.

Quoniam autem rationes propoſitæ $\begin{matrix} AE : CD \\ AB : CF \end{matrix}$ poſſunt eſſe rationes quæcunque $\begin{matrix} \text{majores} \\ \text{minores} \end{matrix}$ ratione $AB : CD$, utcunque parum ab ea diverſæ; tanto magis ratio $X : Z$, ſive major fuerit ratione $AB : CD$, ſive eadem minor, poteſt ad illam accedere, ut ab ea minus diſcrepet, quam alia ratio quæcunque, ſive major ſive minor ratione $AB : CD$. Hinc ratio $AB : CD$ eſt limes (juxta §. 13.) rationis mutabilis $X : Z$.

3°. Sint plures rationes mutabiles ſemper reſpectivè majores totidem rationibus datis; ita ut priores rationes poſſint fieri ſimul minores totidem rationibus

nibus propoſitis, quæ fuerint majores rationibus datis reſpectivè: dico, rationem compoſitam ex rationibus datis eſſe limitem rationis decreſcentis compoſitæ ex rationibus mutabilibus.

Scilicet, ſit v. gr. $A : B < X : Y$ = lim. decr. $X : Y$
$B : C < Y : Z$ = lim. decr. $Y : Z$
$C : D < Z : V$ = lim. decr. $Z : V$
Erit $A : C < X : Z$ = lim. decr. $X : Z$
Atqui $C : D < Z : V$ = lim. decr. $Z : V$
Ergo $A : D < X : V$ = lim. decr. $X : V$.

Et ſi propoſita fuerit quarta ratio tam mutabilis, quam limes hujus rationis decreſcentis; eodem modo demonſtrabitur, rationem ex quatuor rationibus datis compoſitam limitem eſſe rationis decreſcentis compoſitæ ex quatuor rationibus mutabilibus: & ſic deinceps, quicunque ſit numerus rationum propoſitarum.

4°. Sit ratio $A : B$ limes rationis creſcentis $X : Y$
$B : C$ limes rationis decreſcentis $Y : Z$
$C : D$ limes rationis creſcentis $Z : V$.

Dico: rationem $X : V$ (quæ componitur ex rationibus $\left.\begin{matrix} X : Y \\ Y : Z \\ Z : V \end{matrix}\right]$, poſſe fieri $\begin{matrix}\text{majorem}\\\text{minorem}\end{matrix}$, quacunque ratione propoſita, quæ $\begin{matrix}\text{minor}\\\text{major}\end{matrix}$ ſit ratione $A : D$ (quæ componitur ex rationibus $\left.\begin{matrix} A : B \\ B : C \\ C : D \end{matrix}\right]$; ſeu rationem $C : D$ eſſe limitem rationis $X : V$.

Sint AB, BC, CD, DE ipſis Fig. 10.
A, B, C, D, reſpectivè æquales.

1°. Sit ratio propoſita $Ab : CD$ minor ratione $AB : CD$; & proinde ſit $Ab < AB$. Sumatur $Ab' \begin{matrix} > Ab \\ < AB \end{matrix}$; & ſit, uti $Ab : DE$, ita $Ab' : De'$; & proinde ſit $De' > DE$.

Fiat (quod poſſibile per hyp.) $X : Y > Ab' : BC$
Eſt ſemper $Y : Z > BC : CD$
Fiat (quod poſſibile per hyp.) $Z : V > CD : De'$
Erit $X : V > Ab' : De'$
$> Ab : DE$.

2°. Sit ratio propoſita $AB : De$ major ratione $AB : DE$ (& proinde $De < DE$. Fiat, uti $CD : DE$, ita $Cd : De$; & proinde ſit $Cd < CD$.

 Eſt

Eſt ſemper (hyp.) $X : Y < AB : BC$
Fiat (quod poſſibile per hyp.) $Y : Z < BC : Cd$
Et eſt ſemper (hyp.) $Z : V < Cd : De$
Ergo $X : V < AB : De.$

Proinde ratio $X : V$ fieri poteſt $\frac{\text{major}}{\text{minor}}$ quacunque ratione propoſita $\frac{\text{minori}}{\text{majori}}$, quam ratio $AB : DE$. Proinde (§. 13.) ratio $AB : DE$ eſt limes rationis $X : V$.

Methodum demonſtrandi reliquos, qui heic obtinere poſſunt caſus, abunde hoc exemplo liquere exiſtimo.

Obſervatio. Caſus, ubi una aut plures rationes conſtantes occurrunt componendæ cum rationibus mutabilibus (limitum capacibus), ita facilis & evidens eſt ex præcedentibus, ut uno exemplo eum ſufficiat illuſtrare.

Sit ratio $A : B$, limes rationis mutabilis decreſcentis $X : Y$.
Sit ratio $B : C$, æqualis ſemper rationi $Y : Z$.
Dico, rationem $A : C$ eſſe limitem rationis decreſcentis $X : Z$.

Etenim quoniam $X : Y > A : B$
et $Y : Z = B : C$
Erit ſemper $X : Z > A : C$

Sit ratio data $A' : C$ major ratione data $A : C$ $(A' > A)$.

Fiat (quod poſſibile per hyp.) $X : Y < A' : B$
Eſt ſemper $Y : Z = B : C$
Ergo $X : Z < A' : C.$

Proinde ratio $A : C$ compoſita ex rationibus $\begin{matrix} A : B \\ B : C \end{matrix}$ limes eſt rationis decreſcentis $X : Z$.

Corollarium 1. Nominatim, ſit aliqua ratio data limes rationis mutabilis, creſcentis vel decreſcentis; ratio duplicata, triplicata, quadruplicata, - - - - - prioris pariter erit limes rationis creſcentis aut decreſcentis, quæ eſt duplicata, triplicata, quadruplicata - - - - - rationis mutabilis.

Hinc ſpeciatim, ſi ratio quæpiam data limes eſt rationis mutabilis creſcentis vel decreſcentis dimenſionum homologarum duarum figurarum ſimilium, ſuperficialium vel ſolidarum: prior ratio duplicata limes eſt rationis ſuperficierum figurarum; eademque ratio triplicata eſt limes rationis capacitatum figurarum ſolidarum.

Corol-

Corollarium 2. Si rationes, quæ ſunt limites duarum pluriumve rationum mutabilium, reſpectivè æquales ſint rationibus, quæ ſunt limites totidem aliarum rationum mutabilium; dico: rationem, quæ eſt limes rationis ex prioribus rationibus compoſitæ, æqualem eſſe rationi, quæ eſt limes rationis ex poſterioribus compoſitæ.

Exemplum. Sit $\lim. X : Y (= a : b) = \lim. X' : Y'$
$\lim. Y : Z (= b : c) = \lim. Y' : Z'$
$\lim. Z : V (= c : d) = \lim. Z' : V'$, & ſic deinceps
erit $\lim. X : V (= a : d) = \lim. X' : V'$.
Pariter ſit $\lim. X : Y (= a : b) = \lim. X' : Y'$
$Y : Z (= b : c) = \lim. Y' : Z'$
$\lim. Z : V (= c : d) = Z' : V'$
Erit $\lim. X : Y (= a : d) = \lim. X' : Z'$; & ſic deinceps, quicunque ſit rationum numerus, & quomodocunque rationes limites & rationes conſtantes in ſimilibus proportionibus inter ſe combinentur.

Corollarium 3. Si rationis $X : Y$ limes ſit ratio $A : B$
& rationis $Z : V$ limes ſit ratio $C : D$
ſit etiam rationis $XZ : YV$ limes ratio $AC : BD$.

Etenim quoniam $A : B = \lim. X : Y$
$AC : BC = \lim. XZ : YZ$
Pariter $BC : CD = \lim. YZ : YV$
Hinc $AC : CD = \lim. XZ : YV$; quod verum, quicunque ſit rationum numerus.

Corollarium 4. Rationis $A : A \pm x$ limes ſit ratio æqualitatis. Rationis $A \pm x : Y$ limes ſit ratio data $a : b$. Dico, rationis $A \pm x : Y \mp x$ limitem eſſe rationem $a : b$.

Dem. Quoniam $\lim. A : A \pm x = 1 : 1$
et $\lim. A \pm x : Y = a : b$
Erit $\lim. A : Y = a : b$ (§. 14.)
Hinc $\lim. A : A + Y = a : a + b$ (§. 7.)
Sed $\lim. A \pm x : A = 1 : 1$
Ergo $\lim. A \pm x : A + Y = a : a + b$ (§. 14.)
Hinc $\lim. A \pm x : Y \mp x = a : b$ (§. 7.)

§. 16.

§. 15. *Theorema.* Rationis $X : Y$ limes ſit ratio data $a : b$
& rationis $X' : Y'$ limes ſit eadem ratio $a : b$.
Dico, rationis $X+X' : Y+Y'$ limitem eſſe rationem $a : b$.

1°. Ratio data $a : b$, limes ſit utriusque rationis decreſcentis $X : Y$, $X' : Y'$.

Eſt ideo ſemper $X : Y > a : b$
$X' : Y' > a : b$
Proinde $X+X' : Y+Y' > a : b$

Sit ratio propoſita quæcunque $a+\alpha : b$ major ratione $a : b$. Dico, fieri poſſe $X+X' : Y+Y' < a+\alpha : b$.

Etenim, fiat ſimul $X : Y < a+\alpha : b$
$X' : Y' < a+\alpha : b$ (quod fieri poteſt per hyp.)
et erit $X+X' : Y+Y' < a+\alpha : b$.

2°. Demonſtatio eadem eſt, ſi ratio data $a : b$ limes ſit utriusque rationis creſcentis $X : Y$, $X' : Y'$.

3°. Ratio data $a : b$ limes ſit rationis decreſcentis $X : Y$
limes vero rationis creſcentis $X' : Y'$.

Dico, rationem $X+X' : Y+Y'$ poſſe fieri minorem / majorem quacunque ratione data $a+\alpha : b$ majore / $a-\alpha : b$ minore quam ratio $a : b$.

Etenim per ſupp. fieri poteſt $X : Y < a+\alpha : b$
Fiat, & eſt ſemper $X' : Y' < a+\alpha : b$
Ergo erit $X+X' : Y+Y' < a+\alpha : b$.
Item per ſupp. fieri poteſt $X' : Y' > a-\alpha : b$
Fiat, & eſt ſemper $X : Y > a-\alpha : b$
Ergo erit $X+X' : Y+Y' > a-\alpha : b$.

Corollarium. Si rationum quotcunque $X : Y$, $X' : Y'$, $X'' : Y''$, $X''' : Y'''$ ..... limes ſit ratio data $a : b$; erit etiam rationis $X+X'+X''+X'''$ ..... : $Y+Y'+Y''+Y'''$ ..... limes ratio $a : b$.

§. 16. *Theorema.* Sit x quantitas mutabilis, quæ poteſt reddi minor quacunque quantitate propoſita. Sint A, B, C, D .... L, M, N coëfficientes magnitudine dati. Sint etiam a, b, c, d .... l, m, n exponentes poſitivi dati non minores unitate, & ſucceſſive creſcentes. Sit Q functio quantitatis

titatis mutabilis x, qualis ſequitur:

$$Q = Ax^a + Bx^b + Cx^c + Dx^d + \dots Lx^l + Mx^m + Nx^n.$$

Dico, functionem Q poſſe fieri minorem quacunque quantitate propoſita.

Conſtructio. Determinetur quantitas x, ſic, ut quilibet terminus ſeriei propoſitæ inde a primo Ax^a major ſit termino qui illum immediate ſequitur bis ſumtô. Scilicet, ſi duo termini quicunque contigui, ſint Lx^l, Mx^m; fiat $Lx^l > 2Mx^m$; ſeu $x^{m-l} < \frac{L}{2M}$; & fiat x minor quolibet horum valorum ita determinatorum.

Primus Caſus. Omnes coëfficientes A, B, C, $D \dots L$, M, N ſint poſitivi.

Quoniam unusquisque terminus major eſt termino qui illum immediate ſequitur bis ſumtô, quilibet terminus major eſt ſummâ omnium terminorum ſubſequentium, & nominatim, primus terminus Ax^a major eſt ſummâ omnium reliquorum. Proinde $Q < 2Ax^a$. Atqui cum x fieri poſſit minor quacunque quantitate data, a fortiori x^a poteſt reddi minor quacunque quantitate data, & etiam (§. 6.) $2Ax^a$ poteſt reddi minor quacunque quantitate propoſita; & proinde Q poteſt reddi minor quacunque quantitate propoſita.

Secundus Caſus. Coëfficientes dati B, C, D, $E \dots L$, M, N, qui primum ſequuntur, non ſint omnes poſitivi.

Omnibus ut ante factis, tanto magis hoc caſu erit $Q < 2Ax^a$; unde Q poteſt reddi minor quacunque quantitate propoſita.

Quod ſi A ſit quantitas negativa: omnia pariter vera erunt, ſignis mutatis.

Nominatim. Exponentes a, b, c, $d \dots l$, m, n, ſequantur progreſſionem arithmeticam numerorum naturalium; ita ut ſit

$$Q = Ax + Bx^2 + Cx^3 + Dx^4 + \dots Lx^{n-2} + Mx^{n-1} + Nx^n.$$

Facto $Q < 2Ax$, poterit Q eſſe minor quacunque quantitate propoſita. Et quoniam hic caſus eſt omnium frequentiſſimus: ad illum obſervationes ſequentes earumque demonſtrationes applicare ſufficiet.

Obſervatio 1. Quæcunque dicta ſunt de ratione dupla, potuiſſent etiam dici de quavis alia ratione majoris ad minorem. Scilicet, p exiſtente numero majore unitate, ſi ratio cujuslibet termini ſeriei Q ad terminum, qui illum immediate ſequitur, facta fuerit major ratione numeri p ad unitatem, erit

$Q < \frac{p}{p-1} Ax$; & proinde (per hyp. & §. 6.) Q poteſt reddi minor quacunque quantitate propoſita.

Obſervatio 2. Quæ dicta fuerunt de caſu, quo numerus terminorum ſeriei Q datus eſt, applicari etiam, ſub datis conditionibus, poſſunt ad caſum, quo numerus terminorum eſt illimitatus. Quod accurate evolvi meretur, & in Diſſertatione jam memorata *Expoſition elementaire des Principes des Calculs ſuperieurs* haud congrue fuerat omiſſum.

1°. Coëfficientes $A, B, C, D \ldots\ldots$ ſequantur legem aliquam decreſcentem. Fiat $x < \frac{1}{2}$. Erit $x^l > 2x^{l+1}$.

Sed ponitur $L > M$

Ergo a fortiori $Lx^l > 2Mx^{l+1}$. Proinde facto $x < \frac{1}{2}$, omnes termini functionis Q ſequuntur progreſſionem decreſcentem, & quidem velocius quam progreſſio geometrica, cujus quilibet terminus eſt ſubduplus termini præcedentis. Proinde $2Ax$ major eſt ſumma progreſſionis, quicunque ſit numerus terminorum, & Q poteſt reddi minor quacunque quantitate data.

2°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant juxta progreſſionem geometricam quamcunque. Sit ideo $M = pL$

facto $x < \frac{1}{2p}$

erit $x^{l+1} < \frac{1}{2p} x^l$

et $Mx^{l+1} < \frac{1}{2} Lx^l$; ſeu $Lx^l > 2Mx^{l+1}$. Ideo terminus quilibet minor erit duplo præcedentis: unde ut prius concludetur.

3°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant quidem, ſed lentius quam juxta aliquam progreſſionem geometricam.

Ita omnes termini, qui priores duos ſubſequuntur, minores erunt, quam ſi coëfficientes dati creſcerent juxta progreſſionem geometricam, cujus exponens eſſet exponens rationis ſecundi termini ad primum. Unde præcedens concluſio valet a fortiori.

4°. Coëfficientes $A, B, C, D \ldots\ldots$ creſcant quidem a primo inde usque ad coëfficientem datum L, velocius quam in progreſſione geometrica, ſed ab hoc inde (ſi creſcere pergant) lentius quam in progreſſione geometrica creſcant.

Fiat

Fiat (uti 3°.) terminus, cujus coëfficiens L major ſumma omnium terminorum ſequentium. Tum per conſtructionem hujus theorematis fiat primus terminus major ſumma omnium terminorum usque ad terminum, cujus coëfficiens L. Erit, a fortiori, primus terminus major ſumma omnium terminorum ſequentium.

Quod attinet ad caſus, quibus coëfficientes non ſunt omnes poſitivi, aut quibus alterne creſcunt & decreſcunt juxta legem datam, & quod ad incrementa illorum attinet, modo conditionibus præcedentibus conſentaneo: concluſiones illis caſibus a fortiori nectuntur.

Ceteris vero caſibus, quibus coëfficientes ultra omnem limitem creſcentes aliquam legum incrementi præcedentium ſequi non conſtat: hactenus dubito, theorema poſſe ſeriebus, quarum numerus terminorum eſt indeterminatus, applicari.

§. 17. Ut neceſſitas præcedentium diſtinctionum luculentius pateat, exemplum evolvam, quod in ſequentibus magni erit momenti.

Exemplum. Sit x^n potentia quæcunque quantitatis variabilis x, quæ, mutatâ x in $x \pm \Delta x$; fit

$$x^n \pm nx^{n-1}\Delta x + \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2 \pm \frac{n}{1}\dots\frac{n-2}{3}x^{n-3}\Delta x^3 + \frac{n}{1}\dots\frac{n-3}{4}x^{n-4}\Delta x^4 \pm \dots\dots$$

Proinde potentia x^n, mutatâ x in $x \pm \Delta x$, accipit mutationem,

$$\pm\left(nx^{n-1}\Delta x \pm \frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2 + \frac{n}{1}\dots\frac{n-2}{3}x^{n-3}\Delta x^3 \pm \frac{n}{1}\dots\frac{n-3}{4}x^{n-4}\Delta x^4 + \dots\dots\right)$$

Jam vero ponatur, mutationem Δx poſſe fieri minorem quacunque quantitate propoſita: dico, mutationem ſimul factam potentiæ x^n etiam quacunque quantitate propoſita fieri poſſe minorem.

1°. Sit n numerus integer poſitivus. Prior ſeries abrumpitur, & proinde aſſertum verum eſt per §. 16.

2°. Sit n numerus poſitivus non-integer major unitate. Series præcedens non abrumpitur quidem, ſed facile demonſtratur: coëfficientes terminorum, in quibus exponens mutationis Δx ſit major quam $n+1$, continue decreſcere.

Etenim ſit m numerus integer poſitivus immediate minor quam $n + 1$. Coëfficiens termini $x^{n-m}\Delta x^m$, erit $\frac{n}{1}.\frac{n-1}{2}\dots\dots\frac{n-(m-1)}{m}$. Coëfficiens alius

 cujus-

cujuscunque ſequentis termini $n^{n-(m+r)} \Delta x^{m+r}$ eſt

$$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \ldots \cdot \frac{n-m}{m+1} \cdot \frac{n-(m+1)}{m+2} \cdot \ldots \cdot \frac{n-(m+r)}{m+r+1};$$

cujus cum certe unusquisque factor, inde a factore $\frac{n-m}{m+1}$ ſit fractio vera, prædicti coëfficientes continue decreſcunt; præterea iidem alterne fiunt poſitivi & negativi. Proinde (§. 16.) ſeries propoſita poteſt fieri minor quacunque quantitate propoſita.

3°. Sit n numerus negativus quicunque major unitate.

Series præcedens non abrumpitur quidem, ſed exponens rationis duorum coëfficientium ſeſe proxime ſequentium continue decreſcit. Etenim coëfficientes trium terminorum ſeſe proxime ſequentium, ſint

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \cdot \frac{n+m}{m+1}$$

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \ldots \cdot \frac{n+m+1}{m+2}$$

$$\frac{n}{1} \cdot \frac{n+1}{2} \cdot \ldots \ldots \ldots \cdot \frac{n+m+2}{m+3}.$$

Exponens rationis ſecundi horum coëfficentium ad primum, eſt $\frac{n+m+1}{m+2} = 1 + \frac{n-1}{m+2}$;
et exponens rationis coëfficientis tertii ad ſecundum eſt $\frac{n+m+2}{m+3} = 1 + \frac{n-1}{m+3}$:
et quoniam $\frac{n-1}{m+3} < \frac{n-1}{m+2}$, prædicti coëfficientes lentius quam in progreſſione geometrica creſcunt. Sic igitur ſumta ratione $\frac{\Delta x}{x}$, ut primus terminus ſeriei præcedentis major ſit ſecundo ejus termino bis ſumto, tanto magis quilibet alius terminus major erit duplo termini ſequentis; proindeque applicari poteſt demonſtratio §. 16ti.

4°. Exponens n ſit numerus negativus minor unitate.

Singuli factores coëfficientium $\frac{n}{1} \cdot \frac{n+1}{2} \ldots \frac{n+m}{m+1}$, ſunt fractiones veræ, & proinde coëfficientes continue decreſcunt. Quare tanto magis applicari poteſt demonſtratio §. 16ti.

In omni igitur caſu mutatio $\frac{(x+\Delta x)^n - x^n}{x^n - (x-\Delta x)^n}$ poteſt reddi minor quacunque quantitate propoſita.

§. 18.

§. 18. *Theorema.* Omnibus ut in theoremate §. 16. poſitis, ſit $Q = P + Ax^a + Bx^b + Cx^c + Dx^d + \ldots + Lx^l + Mx^m + Nx^n$. Dico, rationem æqualitatis eſſe limitem rationis $Q : P$ decreſcentis aut creſcentis, prouti A eſt quantitas poſitiva aut negativa.

Etenim (per theorema §. 16.) fiat priore caſu $Q < P + 2Ax^a$
altero $Q > P - 2Ax^a$, Ratio æqualitatis eſt limes rationis decreſcentis $P + 2Ax^a : P$
& limes rationis creſcentis $P - 2Ax^a : P$. Ergo, a fortiori, ratio æqualitatis limes eſt rationis decreſcentis aut creſcentis $Q : P$, prouti A eſt quantitas poſitiva aut negativa.

Obſervatio Quæ de caſu, quo numerus terminorum functionis Q eſt illimitatus, præcepta fuerunt (§. 16. Obſerv. 2.) huc pariter valent.

Exemplum. Sit $Q = \frac{(x+\Delta x)^n - x^n}{\Delta x}$, $= \frac{x^n - (x-\Delta x)^n}{\Delta x}$, & mutatio Δx poſſit fieri minor quacunque quantitate propoſita; ratio æqualitatis eſt limes rationis $Q : nx^{n-1}$.

§. 19. *Applicatio.* Sint x & y duæ quantitates, quæ ſimul fieri poſſint quacunque quantitate propoſita minores. Sint Q & Q' duæ functiones prædictarum quantitatum conditionibus §§. 16—18. conſentaneæ, ita ut ſit

$$Q = P + Ax^a + Bx^b + Cx^c + Dx^d + \ldots\ldots$$
$$Q' = P' + A'x^{a'} + B'x^{b'} + C'x^{c'} + D'x^{d'} + \ldots\ldots$$

Dico, limitem $Q : Q' = P : P'$.

Etenim: lim. $Q : P = 1 : 1$ (§. 18.)
$P : P' = P : P'$
lim. $P' : Q' = 1 : 1$ (§. 18.)
Ergo lim. $Q : Q' = P : P'$ (§. 14.)

§. 20. *Theorema.* Sit x quantitas mutabilis, quæ major reddi poteſt quacunque quantitate propoſita. Sint A, B, C, D .... L, M, N coëfficientes magnitudine dati. Sint a, b, c, d .... l, m, n, exponentes etiam dati continue decreſcentes.

Et ſit $Q = Ax^a + Bx^b + Cx^c + Dx^d + \ldots Lx^l + Mx^m + Nx^n$. Dico, rationem æqualitatis eſſe limitem rationis $Q : Ax^a$.

Demonſtratio. $Q = x^a\,(A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$.

Quoniam autem x major poteſt fieri quacunque quantitate propoſita, $\frac{1}{x}$ poteſt fieri minor quacunque quantitate propoſita; & quoniam $a, b, c, d \ldots l, m, n$, continue decreſcunt, exponentes $a-b$, $a-c$, $a-d \ldots a-l$, $a-m$, $a-n$, continue creſcunt; & proinde, quantitate mutabili x poſita majore unitate, termini $\frac{1}{x^{a-b}}, \frac{1}{x^{a-c}}, \frac{1}{x^{a-d}}, \ldots \frac{1}{x^{a-l}}, \frac{1}{x^{a-m}}, \frac{1}{x^{a-n}}$, continue decreſcunt; & proinde (§. 16.) quantitas $B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots + L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$ poteſt reddi minor quacunque quantitate propoſita. Hinc quantitas A limes eſt quantitatis $A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}}$, ſeu

$$\text{lim. } A : A + B\frac{1}{x^{a-b}} + C\frac{1}{x^{a-c}} + D\frac{1}{x^{a-d}} + \ldots L\frac{1}{x^{a-l}} + M\frac{1}{x^{a-m}} + N\frac{1}{x^{a-n}} = 1 : 1$$

& lim. $Ax^a : Ax^a + Bx^b + Cx^c + Dx^d + \ldots Lx^l + Mx^m + Nx^n = 1 : 1$.
ſeu tandem lim. $Ax^a : Q = 1 : 1$.

Exemplum. Exiſtente n numero integro poſitivo: vidimus (§. y'. *Introd.*) ſummam poteſtatum ordinis m numerorum naturalium ab unitate usque ad n continue creſcentium, eſſe ſequentem:

$$\int n^m = \frac{1}{m+1} n^{m+1} + Bn^m + Cn^{m-1} + Dn^{m-2} + En^{m-3} + \ldots\ldots$$

Proinde lim. $\int n^m : \frac{1}{m+1} n^{m+1} = 1 : 1$,

ſeu lim. $\int n^m : n \times n^m = 1 : m+1$.

Hoc eſt: limes rationis ſummæ poteſtatum cujuslibet ordinis numerorum naturalium continue creſcentium inde ab unitate, ad maximum horum numerorum toties ſumptum quot ſunt prædicti numeri, eſt ratio unitatis ad exponentem prædictarum poteſtatum unitate auctum.

Sic lim. $\int n : n.n = 1 : 2$

lim. $\int n^2 : n.n^2 = 1 : 3$

lim. $\int n^3 : n.n^3 = 1 : 4$

lim. $\int n^4 : n.n^4 = 1 : 5$

&c. &c.

§. 21.

§. 21. *Theorema.* Sint duæ quantitates datæ, limites duarum quantitatum mutabilium. Multiplicentur in ſe invicem tam quantitates datæ quam quantitates mutabiles. Dico: productum ex priore multiplicatione ortum, eſſe limitem poſterioris producti.

Sint A & B duæ quantitates datæ, & ſint X & Y duæ quantitates mutabiles, quarum limites ſint A & B. Dico: productum AB eſſe limitem producti XY.

Primus Caſus. Sint quantitates A & B limites quantitatum decreſcentium X & Y. Sit ideo

$$\begin{aligned} X &= A+x \\ Y &= B+y \\ \hline XY &= AB+Ay+Bx+xy. \end{aligned}$$

Dico: productum AB eſſe limitem producti (decreſcentis) XY.

Etenim	lim. B	:	$B+y$	$= 1 : 1$	(§. 3.)
Ergo	lim. Bx	:	$Bx+xy$	$= 1 : 1$	(§ 10.)
Pariterque	lim. $AB+Ay+Bx$	:	$AB+Ay+Bx+xy$	$= 1 : 1$	(§. 2.)

Sed per hyp. x & y poſſunt fieri ſimul minores quacunque quantitate propoſita; ergo & Ay, Bx poſſunt fieri ſimul minores quacunque quantitate propoſita (§. 6.), & etiam $Ay+Bx$ poteſt fieri minor quacunque quantitate propoſita (§. 6.). Proinde quantitas AB eſt limes quantitatis $AB+Ay+Bx$ (§. 6.), & eadem quantitas AB eſt limes quantitatis decreſcentis $AB+Ay+Bx+xy$, ſeu XY. Scilicet

	lim. AB	:	$AB+Ay+Bx$	$= 1 : 1$
	lim. $AB+Ay+Bx$	:	XY	$= 1 : 1$
Ergo	lim. AB	:	XY	$= 1 : 1.$

Er proinde AB eſt limes producti (decreſcentis) XY.

Secundus Caſus. Sint quantitates A & B limites quantitatum creſcentium X & Y. Sit

$$\begin{aligned} X &= A-x \\ Y &= B-y \\ \hline XY &= AB-Ay-Bx+xy. \end{aligned}$$

lim. $B : B-y = 1 : 1$ (§. 3.)

Ergo & lim. $Bx : Bx-xy = 1 : 1$ (§. 10.)

Pariterque, lim. $AB-Ay-Bx : AB-Ay-Bx+xy \, (=XY) = 1 : 1.$ (§. 2.)

Sed

Sed ut prius, AB eſt limes quantitatis (creſcentis) $AB - Ay - Bx$.

Hinc lim. $AB : AB-Ay-Bx = 1 : 1$

lim. $AB-Ay-Bx : XY = 1 : 1$

Ergo lim. $AB : XY = 1 : 1$ (§. 14.)

Proinde AB eſt limes producti creſcentis XY.

Tertius Caſus. Sit A limes quantitatis decreſcentis X.

B limes quantitatis creſcentis Y.

Sit ideo $X = A+x$

$Y = B-y$

$XY = AB-Ay+Bx-xy$.

Erit lim. $B : B-y = 1 : 1$

Ergo lim. $Bx : Bx-xy = 1 : 1$.

Et lim. $AB-Ay+Bx : AB-Ay+Bx-xy (=XY) = 1 : 1$.

Sed ut prius AB eſt limes quantitatis (ſive creſcentis ſive decreſcentis) $AB-Ay+Bx$. Hinc lim. $AB : AB-Ay+Bx = 1 : 1$

lim. $AB-Ay+Bx : XY = 1 : 1$

Ergo lim. $AB : XY = 1 : 1$

Et proinde AB eſt limes producti (ſive creſcentis ſive decreſcentis) XY.

Applicatio. Sint $A, B, C, D, \ldots$ quantitates datæ; & ſint totidem quantitates mutabiles $X, Y, Z, V, \ldots$ quarum priores ſint limites. Dico: productum $ABCD \ldots$ eſſe limitem producti $XYZV \ldots$

Etenim A & B ſunt limites quantitatum mutabilium X & Y;

Ergo AB eſt limes producti mutabilis XY

Sed C eſt limes quantitatis mutabilis Z

Ergo ABC eſt limes producti mutabilis XYZ

Sed D eſt limes quantitatis mutabilis V;

Ergo $ABCD$ eſt limes producti mutabilis $XYZV$; & ſic deinceps.

§. 22. *Lemma notum.* In omni proportione geometrica continua ſumma terminorum extremorum major eſt termino medio bis ſumto.

Corollarium 1. In omni proportione geometrica continua; exceſſus quo maximus terminus medium ſuperat, major eſt exceſſu quo terminus medius ſuperat minimum.

Corol-

Corollarium 2. Sit progreſſio geometrica creſcens: differentia duorum terminorum contiguorum creſcit inde a minimo usque ad maximum.

Corollarium 3. Sint duæ progreſſiones creſcentes, una geometrica, altera arithmetica, quæ duos primos terminos habeant communes, & quarum terminorum numerus idem ſit: reliqui termini progreſſionis geometricæ majores ſunt terminis progreſſionis arithmeticæ æquè ab extremis diſtantibus. Idem a fortiori valet, ſi terminus ſecundus progreſſionis geometricæ major ſit termino ſecundo progreſſionis arithmeticæ.

Corollarium 4. Sint duæ progreſſiones creſcentes, una geometrica, altera arithmetica, quæ duos terminos extremos (h. e. primum & ultimum) communes habent, & quorum numerus terminorum idem ſit. Reliqui termini progreſſionis geometricæ minores ſunt terminis progreſſionis arithmeticæ æque ab extremis diſtantibus.

Theorema. Datis duobus terminis extremis progreſſionis geometricæ, numerus terminorum ita poteſt augeri, ut minoris terminorum datorum & termini huic proximi differentia minor ſit qualibet quantitate data (hoc termino dato minore).

Sint P & Q duo termini dati, quorum P minor; & ſit a quantitas data. Sumatur differentia $Q-P$ terminorum Q & P. Repetatur quantitas data a toties, ut ſuperet differentiam $Q-P$; ſit n numerus vicium, quibus repetita fuit. Dividatur differentia $Q-P$ in n partes æquales. Tum inter terminos datos P & Q inſerantur n termini continue geometrice proportionales. Dico factum.

Etenim inter duos terminos P & Q inſerantur totidem termini n continue arithmetice proportionales: ſint p & p' termini ipſi P proximi in progreſſione geometrica & arithmetica.

Erit (per Coroll. 4. præc.) $p < p'$; & ideo $p - P < p' - P$. Atqui (per conſtructionem) $p' - P < a$; ergo a fortiori $p - P < a$.

Corollarium 1. Terminus itaque datus P limes eſt quantitatis mutabilis p; proinde ratio æqualitatis etiam limes eſt rationis creſcentis $P : p$ (§. 3.)

Corollarium 2. Sit q terminus, qui terminum Q immediate præcedit. Quoniam $P : p = q : Q$, ratio æqualitatis etiam limes eſt rationis creſcentis $q : Q$,

 ſeu

ſeu limes rationis decreſcentis $Q : q$ (§. 10.). Proinde quantitas data Q pariter eſt limes quantitatis mutabilis creſcentis q (§. 3.).

Corollarium 3. Quoniam ratio $\frac{P}{Q} = \left(\frac{P}{p}\right)^n = \left(\frac{q}{Q}\right)^n$; $\left(\frac{P}{Q}\right)^{\frac{1}{n}} = \frac{P}{p} = \frac{q}{Q}$: & pariter $\frac{Q}{P} = \left(\frac{p}{P}\right)^n = \left(\frac{Q}{q}\right)^n$; hinc $\left(\frac{Q}{P}\right)^{\frac{1}{n}} = \frac{p}{P} = \frac{Q}{q}$. Proinde aucto n ratio æqualitatis eſt limes rationis ſubmultiplicatæ rationis cujuslibet propoſitæ; ſeu tam lim. $\left(\frac{P}{Q}\right)^{\frac{1}{n}}$ quam lim. $\left(\frac{Q}{P}\right)^{\frac{1}{n}} = 1$. Quare, ſi $\frac{P}{Q}$ vel $\frac{Q}{P} = z$; lim. $z^{\frac{1}{n}} = 1$, quæcunque ſit z ſive major unitate, ſive minor eâdem.

Scholium. Breviter adhuc, quæ illuſtrandis tum præcedentibus quibusdam, tum ſequentibus neceſſaria ſunt, de quantitatum, quæ majores aut minores fieri poſſunt qualibet quantitate propoſita, poteſtatibus exponentis dati adjungam.

1°. Sit z quantitas, quæ major fieri poteſt qualibet quantitate propoſita; & ſit n exponens datus integer poſitivus. Creſcente z, creſcit a fortiore poteſtas z^n; & proinde, a fortiori, poteſtas z^n major fieri poteſt quacunque quantitate propoſita.

Sit autem exponens datus numerus fractus formæ $\frac{1}{n}$; ſeu proponatur functio $\sqrt[n]{z}$. Quæcunque ſit quantitas propoſita a, fiat $b \geqq a^n$; tum fiat (quod poſſibile per hyp.) $z > b > a^n$: erit etiam $\sqrt[n]{z} > \sqrt[n]{b} > a$.

Hoc pariter obtinet, ſi exponens datus fuerit numerus fractus $\frac{m}{n}$; ita ut functio propoſita ſit $\sqrt[n]{z^m} = z^{\frac{m}{n}}$. Fiat enim $b^m \geqq a^n$; tum fiat $z > b$, & proinde $z^m > b^m$: erit etiam $z^{\frac{m}{n}} > b^{\frac{m}{n}} > a$.

Si autem exponens datus fuerit numerus negativus; ſeu ſi functio propoſita fuerit $\frac{1}{z^n}$: creſcente z, functio hæc minor reddi poteſt quacunque quantitate propoſita. Etenim propoſita ſit quantitas $\frac{1}{a}$: fiat per caſum primum $z^n > a$; & erit $\frac{1}{z^n} < \frac{1}{a}$.

2°. Sit

2°. Sit quantitas z talis, quæ minor reddi poteſt quacunque quantitate propoſita; dico, etiam quamlibet ejus poteſtatem z^n poſſe quacunque quantitate propoſita fieri minorem, ſi n fuerit numerus poſitivus; & contra, eandem quantitatem fieri poſſe quacunque quantitate propoſita majorem, ſi n fuerit numerus negativus.

Caſus hic ad primum facile reducitur. Quoniam enim z poteſt quacunque quantitate propoſita fieri minor; $\frac{1}{z}$ & proinde etiam $\left(\frac{1}{z}\right)^n$ major fieri poteſt quacunque quantitate propoſita; unde z^n minor erit quacunque quantitate propoſita.

Caput Secundum.

De rationibus differentialibus et integralibus.

§. 23.

Sit x^n potentia quæcunque quantitatis variabilis x. Sit etiam $a^{n-1}x$, productum ex eadem quantitate mutabili per quantitatem datam a^{n-1}. Quantitas variabilis x recipiat mutationem quamcunque Δx; unde duæ quantitates $a^{n-1}x$ & x^n fiant $a^{n-1}(x+\Delta x)$ & $(x+\Delta x)^n$; & proinde prædictæ quantitates recipiant mutationes ſimultaneas

$$a^{n-1}\Delta x \ \& \ \frac{n}{1}x^{n-1}\Delta x+\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x^2+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^3+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^4+\ldots$$

Ratio harum mutationum ſimultanearum æqualis eſt ideo rationi

$$a^{n-1} \ : \ \frac{n}{1}x^{n-1}+\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots$$

Proinde, quamdiu n eſt numerus quicunque ab unitate diverſus; ratio prædictarum mutationum differt a ratione $a^{n-1} : nx^{n-1}$. At vero x manente eadem, decreſcente Δx, quantitas $\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots$ poteſt fieri minor quacunque quantitate propoſita (§. 17.); proinde quantitas $\frac{n}{1}x^{n-1}$ limes eſt quantitatis

$$\frac{n}{1}.x^{n-1}+\frac{n}{1}.\frac{n-1}{2}x^{n-2}\Delta x+\frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}\Delta x^2+\frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}\Delta x^3+\ldots\ldots$$

Et ratio $a^{n-1} : \frac{n}{1}x^{n-1}$ limes eſt rationis mutabilis

a^{n-1}

$$a^{n-1} : \frac{n}{1} x^{n-1} + \frac{n}{1} . \frac{n-1}{2} x^{n-2} \Delta x + \frac{n}{1} \ldots \frac{n-2}{3} x^{n-3} \Delta x^2 + \frac{n}{1} \ldots \frac{n-3}{4} x^{n-4} \Delta x^3 + \ldots\ldots$$

ſeu lim. $\frac{\Delta . x^n}{a^{n-1} \Delta x} = \frac{n x^{n-1}}{a^{n-1}}$; & proinde etiam lim. $\frac{\Delta . x^n}{\Delta . x} = n x^{n-1}$.

Quoniam autem hic notandi modus lim. $\frac{\Delta . x^n}{a^{n-1} \Delta x}$ ſeu lim. $\frac{\Delta . x^n}{\Delta . x}$, calculo minus eſt commodus; & tamen hoc ſignum aliaque ſimilia ſæpiſſime occurrunt: facilitatis calculi gratia (& quidem, quod probe notandum, unice hac de cauſa), ſigno minus commodo lim. $\frac{\Delta . x^n}{\Delta x}$ ſubſtituatur hoc $\frac{d . x^n}{d x}$; & ſic conſtituatur æquatio $\frac{d . x^n}{d x} = n x^{n-1}$.

§. 24. Maximopere vero cavendum eſt, ne credamus ſymbolum $\frac{d . x^n}{d x}$, quod formam magnitudinis ex duabus compoſitæ præ ſe fert, revera eſſe ſymbolum compoſitum; ac deſignare fractionem, cujus termini ſint $d . x^n$ & $d x$; quaſi $d . x^n$ & $d x$ denotent certas quantitates, & ut ita dicam, minuſcula (miniatures) quantitatum verarum $\Delta . x^n$ & Δx: aut ne credamus, ex æquatione $\frac{d . x^n}{d x} = n x^{n-1}$ poſſe deduci hanc $d . x^n = n x^{n-1} d x$. Expreſſio $\frac{d . x^n}{d x}$ incomplexa eſt atque peculiaris, ad deſignandos exponentes limitum rationum ſimultaneorum quantitatum mutabilium x^n & x incrementorum facilitatis cauſa introducta: & quamvis veſtigia conſervet originis ſuæ $\frac{\Delta . x^n}{\Delta x}$; ſigna $d . x^n$ & $d x$ ſigillatim ſpectata nullam amplius ad quantitates veras $\Delta . x^n$ & Δx relationem habere firmiter tenendum eſt. Quæ monita eo magis urgenda eſſe cenſeo, quod doctrina hæc quæſtionibus nonnullis inanibus fuit exagitata; quæ ne motæ quidem fuiſſent, ſi ad genuinam ſymboli illius indolem Mathematici ſemper attendiſſent. (a)

Hoc exemplo præeunte, quæ ſequuntur definitiones, facile intelligentur.

§. 24.

(*a*) Nonnulli etiam, qui veram ſigni hujus naturam agnoviſſe videntur, auſi non ſunt, quam hic profiteor ſententiam ampleƈti. Videatur v. gr. Diſſertatio Abbatis Caluso in *novis Commentariis Academiæ Taurinenſis* T. II. ubi ſic ſtatuit Auctor: *Il ne ſuffit pas de voir clairement que* $\frac{d v}{d x}$ *etant la limite du raport* $\frac{\Delta v}{\Delta x}$; *et* $\frac{d z}{d x}$ *la limite du raport* $\frac{\Delta z}{\Delta x}$; *on aura*, $\frac{d v}{d x} = z + x \frac{d z}{d x}$ *lorsque* $v = x z$: *mais, il faut*

d'abord

§. 25. *Definitiones.* Limes rationis mutationum, quas duæ pluresve quantitates mutabiles ſimul ſuſcipiunt, dicatur earum *ratio differentialis;* & exponens hujus rationis dicatur *exponens differentialis.* Operatio, qua exponens differentialis quæritur, dicatur *differentiatio.* Item: calculus, qui occupatur rationibus differentialibus inveſtigandis, dicatur *calculus differentialis.*

Sit P functio quantitatis mutabilis x; exponens differentialis harum quantitatum deſignetur $\frac{dP}{dx}$. Item, ſint P & Q duæ quantitates mutabiles quæcunque, exponens differentialis harum quantitatum deſignetur $\frac{dP}{dQ}$. Et ſymbola $\frac{dP}{dx}$, $\frac{dP}{dQ}$ . . . . ſpectentur tanquam ſigna ſimplicia, nulla apparentis ipſorum compoſitionis ratione habita.

Hinc, ſi P eſt potentia ordinis cujuscunque n quantitatis mutabilis x, $\frac{dP}{dx} = nx^{n-1}$ (§. 23.) Pariter ſit Q potentia alterius ordinis m quantitatis mutabilis x; erit $\frac{dQ}{dx} = mx^{m-1}$. Hinc $\frac{dP}{dQ} = \frac{n}{m}x^{n-m}$.

Etenim $\lim. \Delta P : \Delta x = nx^{n-1} : 1$

item $\lim. \Delta x : \Delta Q = 1 : mx^{m-1}$

Ergo $\lim. \Delta P : \Delta Q = nx^{n-1} : mx^{m-1}$ (§. 14.)

ſeu $\frac{dP}{dQ} = \frac{n}{m}x^{n-m}$.

Item: ſit P functio quantitatis mutabilis x, qualis ſequitur

$$P = Ax^a + Bx^b + Cx^c + Dx^d + \ldots + Lx^l + Mx^m + Nx^n$$

fit $\frac{dP}{dx} = Aax^{a-1} + Bbx^{b-1} + Ccx^{c-1} + Ddx^{d-1} + \ldots + Llx^{l-1} + Mmx^{m-1} + Nnx^{n-1}$.

§. 26. Exercitii cauſa adjungam nonnulla exempla variarum functionum. Sint P & Q duæ quantitates mutabiles, quæ referantur v. gr. ad eandem quantitatem mutabilem x: quæritur exponens differentialis $\frac{d. PQ}{dx}$.

 $(P+\Delta P)$

d'abord attacher une idée nette et preciſe à dv, dx, dz; *afin que ces expreſſions ſignifient quelque choſe par elles-mêmes et independamment les unes des autres.* Cum nuperrime vir adeo ſagax ejusmodi difficultatem moverit, e re profecto eſſe cenſendum eſt, in limine, & cum prima illius ſigni introductione, lectorem de vera ejus ſignificatione monere.

$$(P+\Delta P)(Q+\Delta Q) = PQ + P\Delta Q + Q\Delta P + \Delta P\,.\,\Delta Q$$

$$\text{Hinc} \quad \Delta\,.\,PQ = P\Delta Q + Q\Delta P + \Delta P\,.\,\Delta Q$$

$$\text{et} \quad \frac{\Delta\,.\,PQ}{\Delta x} = P\frac{\Delta Q}{\Delta x} + Q\frac{\Delta P}{\Delta x} + \frac{\Delta P}{\Delta x}\,.\,\Delta Q.$$

$$\text{Hinc} \quad \lim.\frac{\Delta\,.\,PQ}{\Delta x} = P\,.\lim.\frac{\Delta Q}{\Delta x} + Q\lim.\frac{\Delta P}{\Delta x},$$

$$\text{ſeu} \quad \frac{\mathrm{d}\,PQ}{\mathrm{d}\,x} = P\frac{\mathrm{d}\,Q}{\mathrm{d}\,x} + Q\frac{\mathrm{d}\,P}{\mathrm{d}\,x}.$$

$$\text{Eodem modo} \quad \frac{\mathrm{d}.\,PQR}{\mathrm{d}\,x} = PQ\,.\frac{\mathrm{d}R}{\mathrm{d}\,x} + R\frac{\mathrm{d}.\,PQ}{\mathrm{d}\,x}$$

$$= PQ\,\frac{\mathrm{d}R}{\mathrm{d}x} + PR\frac{\mathrm{d}\,Q}{\mathrm{d}\,x} + QR\frac{\mathrm{d}\,P}{\mathrm{d}\,x}$$

$$\frac{\mathrm{d}.\,PQRS}{\mathrm{d}\,x} = PQR\frac{\mathrm{d}\,S}{\mathrm{d}\,x} + S\frac{\mathrm{d}.\,PQR}{\mathrm{d}\,x}$$

$$= PQR\frac{\mathrm{d}\,S}{\mathrm{d}\,x} + PQS\,.\frac{\mathrm{d}R}{\mathrm{d}\,x} + PRS\frac{\mathrm{d}\,Q}{\mathrm{d}\,x} + QRS\frac{\mathrm{d}\,P}{\mathrm{d}\,x}.$$

Et ſic deinceps. Scilicet exponens differentialis producti quotcunque quantitatum mutabilium, & alius cujuscunque quantitatis mutabilis x, æqualis eſt ſummæ productorum omnium priorum quantitatum una excepta, per exponentem differentialem hujus quantitatis & prædictæ quantitatis x.

Eademque regula applicatur ad inveſtigandos exponentes differentiales quantitatum fractarum.

Etenim $\frac{P}{Q} = PQ^{-1}$; ideo $\frac{\mathrm{d}.\frac{P}{Q}}{\mathrm{d}\,x} = P\frac{\mathrm{d}\,Q^{-1}}{\mathrm{d}\,x} + Q^{-1}\frac{\mathrm{d}\,P}{\mathrm{d}\,x}$. Sed (§. 25.) $\frac{\mathrm{d}\,Q^{-1}}{\mathrm{d}\,x} = -Q^{-2}\frac{\mathrm{d}\,Q}{\mathrm{d}\,x}$; & ideo $\frac{\mathrm{d}.\frac{P}{Q}}{\mathrm{d}\,x} = Q^{-1}\frac{\mathrm{d}\,P}{\mathrm{d}\,x} - PQ^{-2}\frac{\mathrm{d}Q}{\mathrm{d}\,x} = \frac{Q\frac{\mathrm{d}\,P}{\mathrm{d}\,x} - P\frac{\mathrm{d}Q}{\mathrm{d}x}}{QQ}$.

Exponens hic differentialis obtinetur etiam modo ſequenti.

Sit $\frac{P}{Q} = Z$, erit $P = QZ$; hinc $\frac{\mathrm{d}P}{\mathrm{d}\,x} = Q\frac{\mathrm{d}Z}{\mathrm{d}\,x} + Z\frac{\mathrm{d}Q}{\mathrm{d}\,x}$

$$= Q\frac{\mathrm{d}Z}{\mathrm{d}\,x} + \frac{P}{Q}\frac{\mathrm{d}Q}{\mathrm{d}\,x}.$$

$$\frac{\mathrm{d}Z}{\mathrm{d}\,x} = \frac{1}{Q}\frac{\mathrm{d}P}{\mathrm{d}\,x} - \frac{P}{QQ}\frac{\mathrm{d}Q}{\mathrm{d}\,x}$$

$$= \frac{Q\frac{\mathrm{d}P}{\mathrm{d}\,x} - P\frac{\mathrm{d}Q}{\mathrm{d}\,x}}{QQ}.$$

Hoc

Hoc eſt: multiplicetur denominator fractionis per exponentem differentialem numeratoris, ab hoc facto ſubtrahatur productum numeratoris per exponentem differentialem denominatoris; oritur exponens differentialis fractionis.

Hisce exemplis continetur univerſus calculus differentialis, quatenus ad functiones tantum algebraicas pertinet. (De quantitatibus tranſcendentibus & exponentialibus poſtea dicemus.)

Sit v. gr. quantitas $\sqrt[3]{(axx-x^3)}$, cujus quæritur exponens differentialis.

Sit $P = \sqrt[3]{(axx-x^3)} = (axx-x^3)^{\frac{1}{3}}$.

$$\frac{dP}{dx} = \tfrac{1}{3}(axx-x^3)^{-\frac{2}{3}}(2ax-3x^2)$$

$$= \frac{2ax-3xx}{3\sqrt[3]{(axx-x^3)^2}} = \frac{2ax-3xx}{3x\sqrt[3]{(x(a-x)^2)}} = \frac{2a-3x}{3\sqrt[3]{(x(a-x)^2)}}.$$

Sit $P = x^m y^n + Ax^p y^q + Bx^r y^s + Cx^t y^v + \ldots\ldots$

$$\frac{dP}{dx} = mx^{m-1}y^n + pAx^{p-1}y^q + rBx^{r-1}y^s + tCx^{t-1}y^v + \ldots\ldots$$

$$+ \frac{dy}{dx}(nx^m y^{n-1} + qAx^p y^{q-1} + sBx^r y^{s-1} + vCx^t y^{v-1} + \ldots\ldots)$$

§. 27. Quemadmodum ex data relatione mutua duarum pluriumve quantitatum mutabilium quæritur earum ratio differentialis: ſic viciſſim ex data ratione differentiali duarum pluriumve quantitatum mutabilium, quæri poteſt relatio mutua ipſarum quantitatum. Hæc relatio dicatur *ratio integralis;* operatio qua ratio integralis quæritur, dicatur *integratio;* & calculus, qui occupatur inveſtigatione rationum integralium, dicatur *calculus integralis.*

Exempla. Data ſit ratio differentialis $\frac{dP}{dx} = x^n = \frac{1}{n+1}(n+1)x^n$: ſit ratio integralis $P = \frac{1}{n+1}x^{n+1}$.

Sit ratio differentialis $\frac{dP}{dx} = y + x\frac{dy}{dx}$; ſit ratio integralis $P = xy$.

Sit ratio differentialis $\frac{dP}{dx} = \frac{y - x\frac{dy}{dx}}{yy}$; ſit ratio integralis $P = \frac{x}{y}$.

Obſervatio 1. Data relatione mutua duarum pluriumve quantitatum mutabilium complexarum, quarum termini partim ſunt conſtantes: hi termini conſtantes non afficiunt exponentem differentialem. Quare viciſſim, data ratione

diffe-

differentiali duarum pluriumve quantitatum mutabilium, non unica ei reſpondet ratio integralis; ſed hæc ratio, præter quantitates mutabiles per rationem differentialem determinatas, aſſumere etiam poteſt terminum conſtantem, qui vulgo per C deſignatur.

Ita ſi $\frac{dP}{dx} = x^n$; fit $P = C + \frac{1}{n+1}x^{n+1}$

$$\frac{dP}{dx} = y + x\frac{dy}{dx}; \quad P = C + xy$$

$$\frac{dP}{dx} = \frac{y - x\frac{dy}{dx}}{yy}; \quad P = C + \frac{x}{y}.$$

Quantitas hæc conſtans C, quæ, rationis differentialis tantum ratione habita, eſt indeterminata, plerumque per naturam quæſtionis propoſitæ determinatur.

Exemplum. Sit $\frac{dP}{dx} = (x-a)^n$; & quæratur ratio integralis huic rationi differentiali reſpondens, talis, ut ratio illa evaneſcat, quando $x = a$.

Ideo $P = C + \frac{1}{n+1}(x-a)^{n+1}$. Atqui quantitas $\frac{1}{n+1}(x-a)^{n+1}$ evaneſcit, poſito $x = a$; ergo $C = 0$: & $P = \frac{1}{n+1}(x-a)^{n+1}$, quæ nunc dicitur ratio integralis completa.

Sit vero $\frac{dP}{dx} = (a+x)^n$; & quæratur ratio integralis talis, ut evaneſcat poſito $x = 0$.

$P = C + \frac{1}{n+1}(a+x)^{n+1}$. Atqui, poſito $x = 0$, $(a+x)^{n+1}$ fit a^{n+1}. Ergo $C = -\frac{1}{n+1}a^{n+1}$. Proinde $P = \frac{1}{n+1}\left((a+x)^{n+1} - a^{n+1}\right)$

$$= \frac{1}{n+1}\left(\frac{n+1}{1}a^n x + \frac{n+1}{1}\cdot\frac{n}{2}a^{n-1}x^2 + \frac{n+1}{1}\ldots\frac{n-1}{3}a^{n-2}x^3 + \frac{n+1}{1}\ldots\frac{n-2}{4}a^{n-3}x^4 + \ldots\ldots\right.$$

$$= a^n x + \frac{n}{1.2}a^{n-1}x^2 + \frac{n.n-1}{1.2.3}a^{n-2}x^3 + \frac{n\ldots n-2}{1\ldots 4}a^{n-3}x^4 + \ldots\ldots$$

Exemplum. Sit $\frac{dP}{dx} = \frac{1}{a+x} = (a+x)^{-1}$; & ſit $P = 0$, quando $x = 0$.

$$P = \frac{x}{a} - \frac{1}{2}\frac{x^2}{a^2} + \frac{1}{3}\frac{x^3}{a^3} - \frac{1}{4}\frac{x^4}{a^4} + \frac{1}{5}\frac{x^5}{a^5} - \frac{1}{6}\frac{x^6}{a^6} + \ldots\ldots$$

Scholium,

Scholium. Sic exceptionem tolli cenſeo communiter huc usque præceptam integrationis formulæ $\frac{dP}{dx} = \frac{1}{a+x} = (a+x)^{-1}$, quæ deducere videtur ad relationem integralem $P = C + \frac{1}{0}(a+x)^{0}$, ſenſu omnino caſſam, & quam ideo alia methodo explicare mathematici allaborarunt: cum oſtenderim, formulam illam eodem prorſus modo evolvi, quo formula generalis $\frac{dP}{dx} = (a+x)^{n}$, ſeu $P = C + \frac{1}{n+1}(a+x)^{n+1}$; addita conditione, quod evaneſcat, ſi $x = 0$. (Vid. præter alios, EULERI *Calculus integralis*, T. I. pag. 26. & 27. ubi illuſtris hic Mathematicus iterata vice exceptionem illam affirmat.) Ipſe eandem ſententiam profeſſus fui in Diſſertatione: *Expoſition elementaire des Principes des Calculs ſuperieurs* pag. 93. Fatendum tamen, caſum hunc a reliquis eatenus eſſe diſtinguendum, quod ſubſtractio indicata $(a+x)^{n+1} - a^{n+1}$ peragi, & diviſione actû inſtituta factor $n+1$ tolli neceſſario debeat, ut formula intelligibilis emergere poſſit.

Obſervatio 2. Data relatione mutua duarum pluriumve quantitatum mutabilium, ratio differentialis earundem ſemper poteſt obtineri, ideoque calculus differentialis poteſt dici omnimode completus. Idem non valet de calculo integrali. Scilicet data ratione differentiali duarum pluriumve quantitatum mutabilium non ſemper obtineri poteſt relatio earum integralis. Nempe non obſtantibus aſſiduis ſagaciſſimorum mathematicorum ſtudiis calculus integralis adhuc in cunabulis jacet; nec ulla regula omnino generalis præſto eſſe dici poteſt, quâ omnes omnino formulæ poſſint ad integrationem perduci. (Quidquid hoc reſpectu ſperaverit ſagaciſs. Dn. PACCASSI. Vide *Phyſikaliſche Arbeiten der einträchtigen Freunde zu Wien.* 1786.)

Nimium a ſcopo meo abluderet, formulas, quarum integrationes in promtu ſunt, exponere, pariter atque artificia, quorum ſubſidio plurimæ formulæ integrantur, quæ primas calculi integralis regulas effugere videntur, tum & inventa varia, quibus integratio formularum complexarum ad integrationem formularum (ſpecie ſaltem) ſimplicium reducuntur. Conſulendi eam in rem ſunt auctores, qui de calculo integrali ſcripſerunt, eo fine, ut ipſum perficerent aut tironibus explanarent: quos inter nominare ſufficiet COTESII Tractatum *de*

Harmonia menſurarum, aut WALMESLEY *Analyſe de la meſure des raports et des angles*, & præ omnibus EULERI *Inſtitutiones calculi integralis.* (*a*)

§. 28. Cum exponentes differentiales quantitatum mutabilium plerumque & ipſi quantitates mutabiles ſint; quæ de qualibet quantitate mutabili dicta ſunt, ejusmodi quoque exponentibus differentialibus poſſunt applicari. Nominatim quæri poſſunt exponentes differentiales iſtiusmodi exponentium differentialium.

Sit v. gr. P functio quantitatis mutabilis x, cujus exponens differentialis $\frac{dP}{dx}$. Quod ſi exponens hic & ipſe mutabilis eſt: indagari pariter poteſt tam ratio mutationum ſimultanearum quantitatum variabilium $\frac{dP}{dx}$ & x; quam ratio, quæ eſt limes iſtius rationis, atque ejus exponens. Ratio, quæ limes eſt rationis mutationum ſimultanearum exponentis differentialis duarum pluriumve quantitatum mutabilium & unius ex illis, dicatur *ratio differentialis ſecundi ordinis;* & exponens prædictæ rationis dicatur *exponens differentialis ſecundi ordinis.*

Exponens itaque differentialis ſecundi ordinis functionis P & quantitatis mutabilis x eſt lim. $\frac{\Delta\frac{dP}{dx}}{\Delta x}$; & eadem de cauſa, quâ prius, nempe majoris calculi facilitatis cauſa, exponens iſte denotetur ſigno $\frac{ddP}{dx^2}$. Et hîc (imo etiam, ſi fieri poſſet, a fortiori) repetenda ſunt, quæ de ſimplicitate ſymboli hujus monui, non obſtante ejus apparenti compoſitione. Ne quis quærat: quid ſit ddP, neque quid ſit dx^2? Nullum, quod ſatisfaciat, obtinebit reſponſum. Conſideret potius ſymbolum $\frac{ddP}{dx^2}$ tanquam expreſſionem unicam & ſui generis, cujus termini apparentes nullam ſervant relationem aut nexum cum quantitatibus $\Delta^2 P$, & Δx^2, ex quibus incautiores mathematici eos derivare aggredi poſſent, & reipſa nimium frequenter aggreſſi ſunt.

Si

(*a*) Utut imperfectus etiamnum ſit calculus integralis: ea tamen eſt multitudo formularum, tam quas integrare, quam quas ad ſimpliciores reducere licet, & formulæ hæ in tot voluminibus, vel a privatis mathematicis, vel ab academiis editis, diſperſæ ſunt; ut gratam æque ac utilem mathematicis navaret operam, qui formulas jam integratas (COTESII & WALMESLEY exemplo), in tabulas apte diſpoſitas redigeret, quarum ope ſæpenumero a labore tædioſo & ſuperfluo vacare liceret.

Si exponens differentialis ſecundi ordinis $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$ ſit & ipſe quantitas mutabilis: operationes, quæ cuilibet quantitati mutabili applicatæ fuerunt, huic etiam exponenti poterunt applicari; & nominatim quæri poterit tam ratio mutationum ſimultanearum quantitatum mutabilium $\frac{\mathrm{dd}P}{\mathrm{d}x^2}$ & x, quam ratio limes iſtius rationis, & ejus exponens. Hæc ratio dicatur *ratio differentialis tertii ordinis*, & exponens ejusdem dicatur *exponens differentialis tertii ordinis*.

Exponens igitur differentialis tertii ordinis prædictæ functionis P eſt $\lim. \frac{\Delta \frac{\mathrm{dd}P}{\mathrm{d}x^2}}{\Delta x}$; & majoris calculi facilitatis cauſa deſignatur ſymbolo $\frac{\mathrm{d}^3 P}{\mathrm{d}x^3}$, quod pariter ſpectandum eſt tanquam ſignum unicum, incomplexum, non quaſi compoſitum ex duabus partibus $\mathrm{d}^3 P$ & $\mathrm{d}x^3$. Neque movenda eſt quæſtio abſona, quid ſint termini illi apparentes; neque comparatio aliqua ſignorum $\mathrm{d}^3 P$ & $\mathrm{d}x^3$ cum quantitatibus $\Delta^3 P$ & Δx^3 eſt inſtituenda.

Hinc facile liquet, quid ſint rationes & exponentes differentiales altiorum ordinum, 4^{ti}, 5^{ti}, $6^{\text{ti}} \ldots n^{\text{ti}}$; & quid denotare cenſenda ſint ſigna illis reſpondentia, $\frac{\mathrm{d}^4 P}{\mathrm{d}x^4}$, $\frac{\mathrm{d}^5 P}{\mathrm{d}x^5}$, $\frac{\mathrm{d}^6 P}{\mathrm{d}x^6} \ldots \frac{\mathrm{d}^n P}{\mathrm{d}x^n}$.

En aliqua exempla præcedentibus illuſtrandis apta.

Sit $P = x^n$.

Erunt
$$\frac{\mathrm{d}P}{\mathrm{d}x} = nx^{n-1}$$
$$\frac{\mathrm{dd}P}{\mathrm{d}x^2} = n.n-1.x^{n-2}$$
$$\frac{\mathrm{d}^3 P}{\mathrm{d}x^3} = n.n-1.n-2.x^{n-3}$$
$$\frac{\mathrm{d}^4 P}{\mathrm{d}x^4} = n.n-1 \ldots n-3.x^{n-4}$$
$$\frac{\mathrm{d}^5 P}{\mathrm{d}x^5} = n.n-1 \ldots n-4.x^{n-5}$$
$$\vdots \qquad \vdots$$
$$\frac{\mathrm{d}^m P}{\mathrm{d}x^m} = n.n-1 \ldots n-(m-1)x^{n-m}.$$

Si exponens n fuerit numerus integer poſitivus, ſeries hæc exponentium differentialium abrumpetur.

 Sci-

Scilicet: ſit $m=n$, erit $\frac{d^n P}{dx^n} = n.n-1 \ldots 2.1.x^0 = 1.2.3 \ldots n$

$$\text{et } \frac{d^{n+1} P}{dx^{n+1}} = 0.$$

Nempe exiſtente n numero integro poſitivo: exponens differentialis, cujus ordo per ipſum potentiæ exponentem deſignatur, ſit conſtans; & exponentes differentiales ordinum ſuperiorum evaneſcunt.

In ceteris caſibus, ubi n non eſt numerus integer poſitivus, nunquam pervenitur ad exponentem differentialem conſtantem, nec ad evaneſcentem.

Secundum Exemplum. Sit $P = zy$, & quærantur omnes exponentes differentiales reſpectu ejusdem variabilis x, cujus z & y cenſeantur eſſe functiones.

Sit ideo $P = zy$

$$\frac{dP}{dx} = z\frac{dy}{dx} + y\frac{dz}{dx}.$$

$$\frac{ddP}{dx^2} = z\frac{ddy}{dx^2} + 2\frac{dz}{dx}.\frac{dy}{dx} + y\frac{ddz}{dx^2}$$

$$\frac{d^3P}{dx^3} = z\frac{d^3y}{dx^3} + 3\frac{dz}{dx}.\frac{ddy}{dx^2} + 3\frac{ddz}{dx^2}.\frac{dy}{dx} + y\frac{d^3z}{dx^3}$$

$$\frac{d^4P}{dx^4} = z\frac{d^4y}{dx^4} + 4\frac{dz}{dx}.\frac{d^3y}{dx^3} + 6\frac{ddz}{dx^2}.\frac{ddy}{dx^2} + 4\frac{d^3z}{dx^3}.\frac{dy}{dx} + y\frac{d^4z}{dx^4}$$

$$\frac{d^5P}{dx^5} = z\frac{d^5y}{dx^5} + 5\frac{dz}{dx}.\frac{d^4y}{dx^4} + 10\frac{ddz}{dx^2}.\frac{d^3y}{dx^3} + 10\frac{d^3z}{dx^3}.\frac{ddy}{dx^2} + 5\frac{d^4z}{dx^4}.\frac{dy}{dx} + y\frac{d^5z}{dx^5}.$$

Generatim.

$$\frac{d^mP}{dx^m} = z\frac{d^my}{dx^m} + \frac{m}{1}\frac{dz}{dx}.\frac{d^{m-1}y}{dx^{m-1}} + \frac{m}{1}.\frac{m-1}{2}\frac{ddz}{dx^2}.\frac{d^{m-2}y}{dx^{m-2}} + \ldots + \frac{m}{1}\frac{dy}{dx}.\frac{d^{m-1}z}{dx^{m-1}} + y\frac{d^mz}{dx^m}.$$

Identitas (jam a LEIBNITZIO obſervata, *Miſcell. Berol.* 1710.) coëfficientium terminorum ſucceſſivorum harum formularum cum coëfficientibus formulæ binominalis Newtonianæ neminem attentum effugiet. Ipſeque differentiationum ſucceſſivarum proceſſus identitatem hanc liquido manifeſtat.

Sit

Sit $P = \frac{y}{x} = yx^{-1}$

$$\frac{dP}{dx} = x^{-1}\frac{dy}{dx} - yx^{-2}$$

$$\frac{ddP}{dx^2} = x^{-1}\frac{ddy}{dx^2} - 2\frac{dy}{dx}x^{-2} + 2yx^{-3}$$

$$\frac{d^3P}{dx^3} = x^{-1}\frac{d^3y}{dx^3} - 3\frac{ddy}{dx^2}x^{-2} + 6\frac{dy}{dx}x^{-3} - 6yx^{-4}$$

$$\frac{d^4P}{dx^4} = x^{-1}\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3}x^{-2} + 12\frac{ddy}{dx^2}x^{-3} - 24\frac{dy}{dx}x^{-4} + 24yx^{-5}.$$

&c. &c. &c.

Sit $P = (xx-aa)^m = (x+a)^m.(x-a)^m.$

Sit $y = (x+a)^m$ $\qquad z = (x-a)^m$

$$\frac{dy}{dx} = m.(x+a)^{m-1} \qquad \frac{dz}{dx} = m(x-a)^{m-1}$$

$$\frac{ddy}{dx^2} = m.m-1.(x+a)^{m-2} \qquad \frac{ddz}{dx^2} = m.m-1.(x-a)^{m-2}$$

$$\frac{d^3y}{dx^3} = m....m-2.(x+a)^{m-3} \qquad \frac{d^3z}{dx^3} = m....m-2.(x-a)^{m-3}$$

$$\frac{d^4y}{dx^4} = m....m-3.(x+a)^{m-4} \qquad \frac{d^4z}{dx^4} = m....m-3.(x-a)^{m-4}$$

$$\frac{d^5y}{dx^5} = m....m-4.(x+a)^{m-5} \qquad \frac{d^5z}{dx^5} = m....m-4.(x-a)^{m-5}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^ny}{dx^n} = m....m-(n-1)(x+a)^{m-n}. \qquad \frac{d^nz}{dx^n} = m....m-(n-1)(x-a)^{m-n}.$$

Hinc $\frac{dP}{dx} = m.(x-a)^m \quad .(x+a)^{m-1}$
$\qquad + m.(x-a)^{m-1}.(x+a)^m$

$$\frac{ddP}{dx^2} = m.m-1.(x-a)^m \quad .(x+a)^{m-2}$$
$$+ 2.m.m.(x-a)^{m-1}.(x+a)^{m-1}$$
$$+ m.m-1.(x-a)^{m-2}.(x+a)^m$$

$$\frac{d^3P}{dx^3} = m.m-1 . m-2 . (x-a)^m . (x+a)^{m-3}$$
$$+ 3.m. m . m-1 . (x-a)^{m-1}. (x+a)^{m-2}$$
$$+ 3.m.m-1 . m . (x-a)^{m-2}. (x+a)^{m-1}$$
$$+ 1.m.m-1.m-2 . (x-a)^{m-3}. (x+a)^m$$

$$\frac{d^4P}{dx^4} = m . . . . . . m-3 . (x-a)^m . (x+a)^{m-4}$$
$$+ 4 . m . m . m-1 . m-2 . (x-a)^{m-1} . (x+a)^{m-3}$$
$$+ 6 . m . m-1 . m . m-1 . (x-a)^{m-2} . (x+a)^{m-2}$$
$$+ 4 . m . m-1 . m-2 . m . (x-a)^{m-3} . (x+a)^{m-1}$$
$$+ 1 . m . . . . . m-3 . (x-a)^{m-4} . (x+a)^m$$

Ordo regularis, juxta quem coëfficientes ſeſe ſubſequuntur, longe apertius patet, quam ſi quantitas $(xx-aa)^m$ non fuiſſet in factores ſuos ſoluta.

Utut regularis ſit poſtrema ſeries; eo tamen caſu, quo m eſt numerus integer negativus, differentiationes ſucceſſivas paulo aliter inſtituere juvabit.

Exempli cauſa: ſit $m = -1$; ideoque $P = \frac{1}{x-a.x+a} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$
$$= \frac{1}{2a}\left((x-a)^{-1} - (x+a)^{-1}\right).$$

Hinc $$\frac{dP}{dx} = \frac{1}{2a}\left\{\begin{matrix} -(x-a)^{-2} \\ +(x+a)^{-2} \end{matrix}\right\}$$
$$\frac{ddP}{dx^2} = \frac{1}{2a}\left\{\begin{matrix} +1.2.(x-a)^{-3} \\ -1.2.(x+a)^{-3} \end{matrix}\right\}$$
$$\frac{d^3P}{dx^3} = \frac{1}{2a}\left\{\begin{matrix} -1.2.3.(x-a)^{-4} \\ +1.2.3.(x+a)^{-4} \end{matrix}\right\}.$$
$$\frac{d^4P}{dx^4} = \frac{1}{2a}\left\{\begin{matrix} +1....4.(x-a)^{-5} \\ -1....4.(x+a)^{-5} \end{matrix}\right\}.$$
$$\frac{d^5P}{dx^5} = \frac{1}{2a}\left\{\begin{matrix} -1....5.(x-a)^{-6} \\ +1....5.(x+a)^{-6} \end{matrix}\right\}$$
&c. &c. &c.

Tironibus identitatem utriusque differentiandi proceſſus oſtendendam, uti varias alias abbreviationes evolvendas, relinquo.

§. 29. Quæcunque de imperfectione calculi integralis, quod ad exponentes differentiales primi gradus attinet, dicta ſunt; tanto magis valent de integratione formularum differentialium ſecundi, tertii, altiorumque ordinum, quæ nonniſi perpaucis caſibus inſtitui poteſt; utcunque eam promovere allaboraverint mathematici, quorum hanc in rem inventa collecta exhibet celeb. EULERUS in opere jam laudato *Inſtitutionum calculi integralis.* Illuſtrationis & exercitii cauſa pauca quædam exempla facillima hic ſufficiant.

Sit $\frac{ddz}{dx^2} = (x+a)^m$

Hinc $\frac{dz}{dx} = C + \frac{1}{m+1}(x+a)^{m+1}$

Unde rurſus $z = C' + Cx + \frac{1}{m+1 \cdot m+2}(x+a)^{m+2}.$

Quoniam quælibet integratio introducit aliquam quantitatem conſtantem, poſt ſecundam integrationem duæ occurrunt quantitates conſtantes C' & C, quæ plerumque per particularem aliquam quæſtionis conditionem determinantur.

Sit

$$\frac{d^n z}{dx^n} = (x+a)^m$$

$$\frac{d^{n-1}z}{dx^{n-1}} = C + \frac{1}{m+1}(x+a)^{m+1}$$

$$\frac{d^{n-2}z}{dx^{n-2}} = C' + Cx + \frac{1}{m+1 \cdot m+2}(x+a)^{m+2}$$

$$\frac{d^{n-3}z}{dx^{n-3}} = C'' + C'x + \tfrac{1}{2}Cx^2 + \frac{1}{m+1 \ldots m+3}(x+a)^{m+3}$$

$$\frac{d^{n-4}z}{dx^{n-4}} = C''' + C''x + \tfrac{1}{2}C'x^2 + \frac{1}{1.2.3}Cx^3 + \frac{1}{m+1 \ldots m+4}(x+a)^{m+4}$$

$$\frac{d^{n-5}z}{dx^{n-5}} = C^{\text{IV}} + C'''x + \frac{1}{1\cdot 2}C''x^2 + \frac{1}{1\ldots 3}C'x^3 + \frac{1}{1\ldots 4}Cx^4 + \frac{1}{m+1 \ldots m+5}(x+a)^{m+5}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{ddz}{dx^2} = C^{N-\text{III}} + C^{N-\text{IV}}x + \frac{1}{1.2}C^{N-\text{V}}x^2 + \;.\;.\;.\;.\;.\; + \frac{1}{1.2\ldots n-3}Cx^{n-3} + \frac{1}{m+1 \ldots m+n-2}(x+a)^{m+n-2}$$

$$\frac{dz}{dx} = C^{N-\text{II}} + C^{N-\text{III}}x + \frac{1}{1.2}C^{N-\text{IV}}x^2 + \frac{1}{1.2.3}C^{N-\text{V}}x^3 + \ldots + \frac{1}{1.2\ldots n-2}Cx^{n-2} + \frac{1}{m+1 \ldots m+n-1}(x+a)^{m+n-1}$$

$$z = C^{N-\text{I}} + C^{N-\text{II}}x + \frac{1}{1.2}C^{N-\text{III}}x^2 + \frac{1}{1.2.3}C^{N-\text{IV}}x^3 + \ldots + \frac{1}{1.2\ldots n-1}Cx^{n-1} + \frac{1}{m+1 \ldots m+n}(x+a)^{m+n}.$$

CAPUT

Caput Tertium.

De theoremate Tayloriano.

§. 30.

Theoremate, quod Taylorianum dicitur, data mutatione quantitatis mutabilis, determinatur mutatio, quam ejusdem quantitatis mutabilis functio quæcunque recipit.

Theorema hoc tam multiplicis eſt in calculo differentiali & integrali uſus, ut accurate explicari, & quo par eſt rigore, demonſtrari mereatur. Taylor (Mathematicus Anglus) illud expoſuit in opere ſuo inſcripto: *Methodus incrementorum directa & inverſa* (Lond. 1715.); ſed ex formula (§. *p. Introd.*)

$$P^m = P + \frac{m}{1}\Delta P + \frac{m}{1}\cdot\frac{m-1}{2}\Delta^2 P + \frac{m}{1}\ \ldots\ \frac{m-2}{3}\Delta^3 P + \ldots\ldots$$

$$= P + \frac{m\Delta x}{1}\cdot\frac{\Delta P}{\Delta x} + \frac{m\Delta x}{1}\cdot\frac{(m-1)\Delta x}{2}\cdot\frac{\Delta^2 P}{\Delta x^2} + \frac{m\Delta x}{1}\cdot\frac{(m-1)\Delta x}{2}\cdot\frac{(m-2)\Delta x}{3}\cdot\frac{\Delta^3 P}{\Delta x^3} + \ldots$$

conſequi tantum indicavit. Plerique poſt illum autores ſupplendæ huic deductioni ideam infiniti adhibuerunt, quam hoc opere penitus eliminare ſtudui.

In Diſſertatione inſcripta: *Expoſition elementaire des principes des calculs ſuperieurs*, theorematis hujus demonſtrationem ad notiones claras & mere elementares revocare allaboravi. Cum vero, quam pag. 47—51. propoſui, demonſtraſtratio (inprimis autem quod attinet ad expreſſiones in pag. 51. contentas), minus firma mihi videretur, neque mihi omnino ſatisfaceret (quod innui pag. 207.); aliam pag. 208—210. ſubjunxi, Academiæ Berolinenſis de prædicta Diſſertatione judicio poſteriorem; quam & ill. Kæstnerum (*Anfangsgründe der Analyſis des Unendlichen, 2te Aufl.* §. 144 ſeq.) jam tradidiſſe poſtea comperi. Nititur ea hoc fundamento, quod functio quælibet quantitatis mutabilis ſerie potentiarum hujus quantitatis in conſtantes quasdam ductarum exprimi poſſit, qua reductione admiſſa, demonſtratio hæc mihi plane ſatisfacit.

§. 31. *Primus Caſus.* Sit x^n poteſtas quæcunque quantitatis mutabilis x, quæ fiat $x+b$; ita ut x^n fiat $(x+b)^n$. Jam vidimus (§. *e. Introd.*) eſſe

$$(x+b)^n = x^n + \frac{n}{1}x^{n-1}b + \frac{n}{1}\cdot\frac{n-1}{2}x^{n-2}b^2 + \frac{n}{1}\ldots\frac{n-2}{3}x^{n-3}b^3 + \frac{n}{1}\ldots\frac{n-3}{4}x^{n-4}b^4 + \ldots$$

Sit

Sit $P = x^n$

Erit (§. 28.) $\frac{dP}{dx} = nx^{n-1}$

$$\frac{ddP}{dx^2} = n.n-1.x^{n-2}$$

$$\frac{d^3P}{dx^3} = n.n-1.n-2.x^{n-3}$$

$$\frac{d^4P}{dx^4} = n \ldots . n-3.x^{n-4}$$

$$\frac{d^5P}{dx^5} = n \ldots . n-4.x^{n-5}$$

&c. &c.

Hinc $(x+b)^n = x^n + \frac{b}{1}\frac{dP}{dx} + \frac{b^2}{1.2}\frac{ddP}{dx^2} + \frac{b^3}{1\ldots3}\frac{d^3P}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4P}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5P}{dx^5} + \ldots\ldots$

Proinde exiſtente **P** potentia qualibet quantitatis mutabilis x, valor hujus poteſtatis, reſpondens mutationi propoſitæ b quantitatis mutabilis x, exprimitur per hanc mutationem b, & per exponentes differentiales ordinum ſucceſſivorum ipſius illius poteſtatis & quantitatis mutabilis x.

§. 32. *Secundus Caſus.* Functio propoſita P quantitatis mutabilis x aut ſit immediate ex potentiis quantitatis mutabilis x compoſita, aut ad eas reducta.

Scilicet ſit $P = Ax^a + Bx^b + Cx^c + Dx^d + Ex^e \ldots\ldots$

ſeu ſit $P = Q + R + S + T + V \ldots\ldots$

ubi Q, R, S, T, $V \ldots\ldots$ ſunt potentiæ quælibet quantitatis mutabilis x, coëfficientibus datis affectæ.

Sint etiam P', Q', R', S', T', $V' \ldots\ldots$ valores omnium prædictarum functionum, quos recipiunt, quando x, accepta mutatione b, fit $x+b$.

Erit ideo $P = Q + R + S + T + V + \ldots\ldots$

$$\frac{dP}{dx} = \frac{dQ}{dx} + \frac{dR}{dx} + \frac{dS}{dx} + \frac{dT}{dx} + \frac{dV}{dx} + \ldots\ldots$$

$$\frac{ddP}{dx^2} = \frac{ddQ}{dx^2} + \frac{ddR}{dx^2} + \frac{ddS}{dx^2} + \frac{ddT}{dx^2} + \frac{ddV}{dx^2} + \ldots\ldots$$

$$\frac{d^3P}{dx^3} = \frac{d^3Q}{dx^3} + \frac{d^3R}{dx^3} + \frac{d^3S}{dx^3} + \frac{d^3T}{dx^3} + \frac{d^3V}{dx^3} + \ldots\ldots$$

$$\frac{d^4P}{dx^4} = \frac{d^4Q}{dx^4} + \frac{d^4R}{dx^4} + \frac{d^4S}{dx^4} + \frac{d^4T}{dx^4} + \frac{d^4V}{dx^4} + \ldots\ldots$$

$$\frac{d^5P}{dx^5} = \frac{d^5Q}{dx^5} + \frac{d^5R}{dx^5} + \frac{d^5S}{dx^5} + \frac{d^5T}{dx^5} + \frac{d^5V}{dx^5} + \ldots\ldots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{d^mP}{dx^m} = \frac{d^mQ}{dx^m} + \frac{d^mR}{dx^m} + \frac{d^mS}{dx^m} + \frac{d^mT}{dx^m} + \frac{d^mV}{dx^m} + \ldots\ldots$$

Item $P' = Q' + R' + S' + T' + V' + \ldots\ldots$

Atqui (*Cas. I.*)

$$Q' = Q + \frac{b}{1}\frac{dQ}{dx} + \frac{b^2}{1.2}\frac{ddQ}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3Q}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4Q}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^4Q}{dx^5} + \ldots\ldots$$

$$R' = R + \frac{b}{1}\frac{dR}{dx} + \frac{b^2}{1.2}\frac{ddR}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3R}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4R}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5R}{dx^5} + \ldots\ldots$$

$$S' = S + \frac{b}{1}\frac{dS}{dx} + \frac{b^2}{1.2}\frac{ddS}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3S}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4S}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5S}{dx^5} + \ldots\ldots$$

$$T' = T + \frac{b}{1}\frac{dT}{dx} + \frac{b^2}{1.2}\frac{ddT}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3T}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4T}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5T}{dx^5} + \ldots\ldots$$

$$V' = V + \frac{b}{1}\frac{dV}{dx} + \frac{b^2}{1.2}\frac{ddV}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3V}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4V}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5V}{dx^5} + \ldots\ldots$$

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Hinc $P' = P + \frac{b}{1}\frac{dP}{dx} + \frac{b^2}{1.2}\frac{ddP}{dx^2} + \frac{b^3}{1.2.3}\frac{d^3P}{dx^3} + \frac{b^4}{1\ldots4}\frac{d^4P}{dx^4} + \frac{b^5}{1\ldots5}\frac{d^5P}{dx^5} + \ldots\ldots$

Proinde, admiſſa functionis cujuslibet quantitatis mutabilis in poteſtates quantitatis hujus decompoſitione, rigoroſe omnino theorema Taylorianum demonſtratur. Deſiderari tamen poterat, ipſum, reductione illa non ſuppoſita, ſufficienter ſtabiliri: quod in Diſſertatione inſcripta *Theorematis Tayloriani demonſtratio* (Tubingæ 1789.) præſtitit clar. PFLEIDERER, poſtulando tantum, mutationem ΔP functionis P, quæ eſt functio mutationis Δx variabilis x, talem eſſe, ut exprimi poſſit per potentias hujus mutationis, quarum exponentes ſequuntur

tur progreſſionem numerorum naturalium. Hujus methodum ſuccinƈte expoſitam hic ſubjungam.

§. 33. Sit P funƈtio quæcunque quantitatis mutabilis x.

Sint P^{I}, P^{II}, P^{III}, P^{IV}, P^{V} . . . . P^{N}, P^{N+1}, valores, quos funƈtio illa induit, ſi loco x ſubſtituantur quantitates

$$x+\Delta x,\ x+2\Delta x,\ x+3\Delta x,\ x+4\Delta x,\ x+5\Delta x \ldots\ldots x+n\Delta x,\ x+(n+1)\Delta x.$$

Sit etiam
$$\begin{array}{lllllll}
P^{I} & = P + & A\Delta x + & B\Delta x^2 & + \ C\Delta x^3 + & D\Delta x^4 & + \ldots\ldots \\
\end{array}$$
Hinc
$$\begin{array}{lllllll}
P^{II} & = P + & 2A\Delta x + & 2^2 B\Delta x^2 & + \ 2^3 C\Delta x^3 + & 2^4 D\Delta x^4 & + \ldots\ldots \\
P^{III} & = P + & 3A\Delta x + & 3^2 B\Delta x^2 & + \ 3^3 C\Delta x^3 + & 3^4 D\Delta x^4 & + \ldots\ldots \\
P^{IV} & = P + & 4A\Delta x + & 4^2 B\Delta x^2 & + \ 4^3 C\Delta x^3 + & 4^4 D\Delta x^4 & + \ldots\ldots \\
P^{V} & = P + & 5A\Delta x + & 5^2 B\Delta x^2 & + \ 5^3 C\Delta x^3 + & 5^4 D\Delta x^4 & + \ldots\ldots \\
\vdots & & & \vdots & & \vdots & \\
P^{N} & = P + & nA\Delta x + & n^2 B\Delta x^2 + & n^3 C\Delta x^3 + & n^4 D\Delta x^4 & + \ldots\ldots \\
P^{N+1} & = P + & (n+1)A\Delta x + & (n+1)^2 B\Delta x^2 + & (n+1)^3 C\Delta x^3 + & (n+1)^4 D\Delta x^4 & + \ldots\ldots
\end{array}$$

Unde

$$\begin{array}{llllll}
\Delta P\,(=P^{I}-P) & = A\Delta x + & B\Delta x^2 + & C\Delta x^3 + & D\Delta x^4 & + \ldots\ldots \\
\Delta P^{I}\,(=P^{II}-P^{I}) & = A\Delta x + & (2^2-1)B\Delta x^2 + & (2^3-1)C\Delta x^3 + & (2^4-1)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{II}\,(=P^{III}-P^{II}) & = A\Delta x + & (3^2-2^2)B\Delta x^2 + & (3^3-2^3)C\Delta x^3 + & (3^4-2^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{III}\,(=P^{IV}-P^{III}) & = A\Delta x + & (4^2-3^2)B\Delta x^2 + & (4^3-3^3)C\Delta x^3 + & (4^4-3^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{V}\,(=P^{V}-P^{IV}) & = A\Delta x + & (5^2-4^2)B\Delta x^2 + & (5^3-4^3)C\Delta x^3 + & (5^4-4^4)D\Delta x^4 & + \ldots\ldots \\
\vdots & & \vdots & & \vdots & \\
\Delta P^{N-1}\,(=P^{N}-P^{N-1}) & = A\Delta x + & (n^2-(n-1)^2)B\Delta x^2 + & (n^3-(n-1)^3)C\Delta x^3 + & (n^4-(n-1)^4)D\Delta x^4 & + \ldots\ldots \\
\Delta P^{N}\,(=P^{N+1}-P^{N}) & = A\Delta x + & ((n+1)^2-n^2)B\Delta x^2 + & ((n+1)^3-n^3)C\Delta x^3 + & ((n+1)^4-n^4)D\Delta x^4 & + \ldots\ldots
\end{array}$$

In his æquationibus coëfficientes τȣ $A\Delta x$ differentiæ ſunt primi ordinis numerorum naturalium; proinde (§. *q. Introd.*) inter ſe æquales, nempe $= 1$; & differentiæ ulteriorum ordinum evaneſcunt. Hinc fit

$$\Delta^2 P(=\Delta P^{\text{I}}-\Delta P) = \Delta^{\text{II}}(2^2\dots 1)B\Delta x^2 + \Delta^{\text{II}}(2^3\dots 1)C\Delta x^3 + \Delta^{\text{II}}(2^4\dots 1)D\Delta x^4 + \dots$$
$$\Delta^2 P^{\text{I}}(=\Delta P^{\text{II}}-\Delta P^{\text{I}}) = \Delta^{\text{II}}(3^2\dots 1^2)B\Delta x^2 + \Delta^{\text{II}}(3^3\dots 1^3)C\Delta x^3 + \Delta^{\text{II}}(3^4\dots 1^4)D\Delta x^4 + \dots$$
$$\Delta^2 P^{\text{II}}(=\Delta P^{\text{III}}-\Delta P^{\text{II}}) = \Delta^{\text{II}}(4^2\dots 2^2)B\Delta x^2 + \Delta^{\text{II}}(4^3\dots 2^3)C\Delta x^3 + \Delta^{\text{II}}(4^4\dots 2^4)D\Delta x^4 + \dots$$
$$\Delta^2 P^{\text{III}}(=\Delta P^{\text{IV}}-\Delta P^{\text{III}}) = \Delta^{\text{II}}(5^2\dots 3^2)B\Delta x^2 + \Delta^{\text{II}}(5^3\dots 3^3)C\Delta x^3 + \Delta^{\text{II}}(5^4\dots 3^4)D\Delta x^4 + \dots$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\Delta^2 P^{\text{N-I}}(=\Delta P^{\text{N}}-\Delta P^{\text{N-I}}) = \Delta^{\text{II}}(n+1)^2..(n-1^2)B\Delta x^2 + \Delta^{\text{II}}(n+1)^3..(n-1)^3 C\Delta x^3 + \Delta^{\text{II}}(n+1)^4..(n-1)^4 D\Delta x^4 + \dots$$
$$\Delta^2 P^{\text{N}}(=\Delta P^{\text{N+I}}-\Delta P^{\text{N}}) = \Delta^{\text{II}}(n+2)^2\dots n^2)Bx^2 + \Delta^{\text{II}}(n+2)^3\dots n^3)C\Delta x^3 + \Delta^{\text{II}}(n+2)^4\dots n^4)D\Delta x^4 + \dots$$

In his æquationibus coëfficientes τȣ $B\Delta x^2$ ſunt differentiæ ſecundi ordinis ſecundarum poteſtatum numerorum naturalium; proinde (§. *q. Introd.*) conſtantes, nempe $= 1.2$; & differentiæ alteriorum ordinum evaneſcunt. Hinc fit

$$\Delta^3 P(=\Delta^2 P^{\text{I}}-\Delta^2 P) = \Delta^{\text{III}}(3^3\dots 1^3)C\Delta x^3 + \Delta^{\text{III}}(3^4\dots 1^4)D\Delta x^4 + \Delta^{\text{III}}(3^5\dots 1^5)E\Delta x^5 + \dots$$
$$\Delta^3 P^{\text{I}}(=\Delta^2 P^{\text{II}}-\Delta^2 P^{\text{I}}) = \Delta^{\text{III}}(4^3\dots 1^3)C\Delta x^3 + \Delta^{\text{III}}(4^4\dots 1^4)D\Delta x^4 + \Delta^{\text{III}}(4^5\dots 1^5)E\Delta x^5 + \dots$$
$$\Delta^3 P^{\text{II}}(=\Delta^2 P^{\text{III}}-\Delta^2 P^{\text{II}}) = \Delta^{\text{III}}(5^3\dots 2^3)C\Delta x^3 + \Delta^{\text{III}}(5^4\dots 2^4)D\Delta x^4 + \Delta^{\text{III}}(5^5\dots 2^5)E\Delta x^5 + \dots$$
$$\Delta^3 P^{\text{III}}(=\Delta^2 P^{\text{IV}}-\Delta^2 P^{\text{III}}) = \Delta^{\text{III}}(6^3\dots 3^3)C\Delta x^3 + \Delta^{\text{III}}(6^4\dots 3^4)D\Delta x^4 + \Delta^{\text{III}}(6^5\dots 3^5)E\Delta x^5 + \dots$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\Delta^3 P^{\text{N-I}}(=\Delta^2 P^{\text{N}}-\Delta^2 P^{\text{N-I}}) = \Delta^{\text{III}}(n+2)^3..n-1)^3 C\Delta x^3 + \Delta^{\text{III}}(n+2)^4..n-1)^4 D\Delta x^4 + \Delta^{\text{III}}(n+2)^5..(n-1)^5 E\Delta x^5 + \dots$$
$$\Delta^3 P^{\text{N}}(=\Delta^2 P^{\text{N+I}}-\Delta^2 P^{\text{IV}}) = \Delta^{\text{III}}(n+3)^3\dots n^3)C\Delta x^3 + \Delta^{\text{III}}(n+3)^4\dots n^4)D\Delta x^4 + \Delta^{\text{III}}(n+3)^5\dots n^5)E\Delta x^5 + \dots$$

In his æquationibus coëfficientes τȣ $C\Delta x^3$ ſunt differentiæ tertii ordinis tertiarum poteſtatum numerorum naturalium; proinde (§. *q. Introd.*) conſtantes, nempe $= 1.2.3$, & differentiæ ulteriorum ordinum illorum coëfficientium evaneſcunt. Hinc rurſus

$$\Delta^4 P(=\Delta^3 P^{\text{I}}-\Delta^3 P) = \Delta^{\text{IV}}(4^4\dots 1)D\Delta x^4 + \Delta^{\text{IV}}(4^5\dots 1)E\Delta x^5 + \Delta^{\text{IV}}(4^6\dots 1)F\Delta x^6 + \dots$$
$$\Delta^4 P^{\text{I}}(=\Delta^3 P^{\text{II}}-\Delta^3 P^{\text{I}}) = \Delta^{\text{IV}}(5^4\dots 1)D\Delta x^4 + \Delta^{\text{IV}}(5^5\dots 1)E\Delta x^5 + \Delta^{\text{IV}}(5^6\dots 1)F\Delta x^6 + \dots$$
$$\Delta^4 P^{\text{II}}(=\Delta^3 P^{\text{III}}-\Delta^3 P^{\text{II}}) = \Delta^{\text{IV}}(6^4\dots 2^4)D\Delta x^4 + \Delta^{\text{IV}}(6^5\dots 2^5)E\Delta x^5 + \Delta^{\text{IV}}(6^6\dots 2^6)F\Delta x^6 + \dots$$
$$\Delta^4 P^{\text{III}}(=\Delta^3 P^{\text{IV}}-\Delta^3 P^{\text{III}}) = \Delta^{\text{IV}}(7^4\dots 3^4)D\Delta x^4 + \Delta^{\text{IV}}(7^5\dots 3^5)E\Delta x^5 + \Delta^{\text{IV}}(7^6\dots 3^6)F\Delta x^6 + \dots$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$\Delta^4 P^{\text{N-I}}(=\Delta^3 P^{\text{N}}-\Delta^3 P^{\text{N-I}}) = \Delta^{\text{IV}}(n+3)^4..(n-1)^4 D\Delta x^4 + \Delta^{\text{IV}}(n+3)^5..(n-1)^5 E\Delta x^5 + \Delta^{\text{IV}}(n+3)^6..(n-1)^6 F\Delta x^6 + \dots$$
$$\Delta^4 P^{\text{N}}(=\Delta^3 P^{\text{N+I}}-\Delta^3 P^{\text{N}}) = \Delta^{\text{IV}}(n+4^4\dots n)^4 D\Delta x^4 + \Delta^{\text{IV}}(n+4)^5\dots n^5)E\Delta x^5 + \Delta^{\text{IV}}(n+4)^6\dots n^6)F\Delta x^6 + \dots$$

In his æquationibus coëfficientes τȣ $D\Delta x^4$ ſunt differentiæ quarti ordinis poteſta-

tum

tum quartarum numerorum naturalium; proinde (§. q. *Introd.*) conſtantes, nempe = 1. 2. 3. 4. & differentiæ ulteriorum ordinum evaneſcunt.

Hinc ſumtis differentiis quinti ordinis functionis P coëfficientes τῦ $E\Delta x^5$ ſiunt differentiæ quinti ordinis poteſtatum quintarum numerorum naturalium; proinde (§. q. *Introd.*) conſtantes, nempe = 1. 2. . . . 5. & differentiæ ulteriorum ordinum evaneſcunt.

Unde procedendo ad differentias ſexti, ſeptimi, octavi . . . . ordinis functionis P; coëfficientes terminorum $F\Delta x^6$, $G\Delta x^7$, $H\Delta x^8$ . . . . qui reſpective ſunt differentiæ - - ſexti, ſeptimi, octavi . . . . ordinis poteſtatum - - - - ſextarum, ſeptimarum, octavarum . . . . numerorum naturalium ſiunt reſpective quantitates conſtantes,

1. 2. . . 6, 1. 2. . . 7, 1. 2. . . 8 . . . . & differentiæ ulteriorum ordinum evaneſcunt.

Sumtis itaque rationibus differentialibus ſucceſſivis, quæ ex præcedentibus æquationibus conſequuntur, ſit

$$\frac{dP}{dx} = A \qquad \text{Unde } A = 1\,\frac{dP}{dx}$$

$$\frac{ddP}{dx^2} = 1.2B \qquad B = \frac{1}{1.2}\frac{ddP}{dx^2}$$

$$\frac{d^3P}{dx^3} = 1.2.3C \qquad C = \frac{1}{1.2.3}\frac{d^3P}{dx^3}$$

$$\frac{d^4P}{dx^4} = 1.2..4D \qquad D = \frac{1}{1.2...4}\frac{d^4P}{dx^4}$$

$$\frac{d^5P}{dx^5} = 1.2...5E \qquad E = \frac{1}{1.2...5}\frac{d^5P}{dx^5}$$

$$\frac{d^6P}{dx^6} = 1.2....6F \qquad F = \frac{1}{1.2...6}\frac{d^6P}{dx^6}$$

$$\vdots \qquad \vdots$$

$$\frac{d^mP}{dx^m} = 1.2....mM \qquad M = \frac{1}{1.2...m}\frac{d^mP}{dx^m}.$$

Pro-

Proinde

$$P^{\mathrm{I}}(=P+A\Delta x+B\Delta x^2+C\Delta x^3+D\Delta x^4+E\Delta x^5+F\Delta x^6+\ldots)$$
$$=P+\frac{\Delta x}{\mathrm{I}}\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{\Delta x^2}{\mathrm{I.\,2}}\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{\Delta x^3}{\mathrm{I.2.3}}\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\frac{\Delta x^4}{\mathrm{I.2..4}}\frac{\mathrm{d}^4P}{\mathrm{d}x^4}+\frac{\Delta x^5}{\mathrm{I.2..5}}\frac{\mathrm{d}^5P}{\mathrm{d}x^5}+\frac{\Delta x^6}{\mathrm{I.2..6}}\frac{\mathrm{d}^6P}{\mathrm{d}x^6}+\ldots$$

§. 34. Uberrimi hujus propoſitionis uſus in ſequentibus patebunt. Heic unum exemplum mere algebraicum utilitatis ipſius afferre ſufficiet: differentias omnium ordinum functionis cujuslibet quantitatis mutabilis per differentias poteſtatum numerorum naturalium determinando.

Quoniam
$$P^{\mathrm{N}}=P+\frac{n}{\mathrm{I}}\Delta x\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{n^2}{\mathrm{I.\,2}}\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{n^3}{\mathrm{I.\,2.\,3}}\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\frac{n^4}{\mathrm{I.2\ldots4}}\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4}+\frac{n^5}{\mathrm{I.2\ldots5}}\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5}+\text{- - - - -}+\text{- - - - -}$$

et
$$P^{\mathrm{N+I}}=P+\frac{n+\mathrm{I}}{\mathrm{I}}\Delta x\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{(n+\mathrm{I})^2}{\mathrm{I.\,2}}\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{(n+\mathrm{I})^3}{\mathrm{I.\,2.\,3}}\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\frac{(n+\mathrm{I})^4}{\mathrm{I.2\ldots4}}\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4}+\frac{(n+\mathrm{I})^5}{\mathrm{I.2\ldots5}}\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5}+\text{- - - - -}+\text{- - - - -}$$

Erit
$$\Delta P^{\mathrm{N}}\,(=P^{\mathrm{N+I}}-P^{\mathrm{N}})=\frac{n+\mathrm{I}-n}{\mathrm{I}}\Delta x\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{(n+\mathrm{I})^2-n^2}{\mathrm{I.\,2}}\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{(n+\mathrm{I})^3-n^3}{\mathrm{I.\,2.\,3}}\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\frac{(n+\mathrm{I})^4-n^4}{\mathrm{I.\,2\ldots4}}\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4}+\frac{(n+\mathrm{I})^5-n^5}{\mathrm{I.\,2\ldots5}}\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5}+\text{- - - - - -}+\text{- - - - - -}$$

$$\Delta P^{\mathrm{N+I}}=\frac{n+2-(n+\mathrm{I})}{\mathrm{I}}\Delta x\frac{\mathrm{d}P}{\mathrm{d}x}+\frac{(n+2)^2-(n+\mathrm{I})^2}{\mathrm{I.\,2}}\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2}+\frac{(n+2)^3-(n+\mathrm{I})^3}{\mathrm{I.\,2.\,3}}\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3}+\frac{(n+2)^4-(n+\mathrm{I})^4}{\mathrm{I.\,2\ldots4}}\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4}+\frac{(n+2)^5-(n+\mathrm{I})^5}{\mathrm{I.\,2\ldots5}}\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5}+\text{- - - - - - -}+\text{- - - - - - -}$$

Hinc

Hinc $\Delta^2 P^{N} (= \Delta P^{N+1} - \Delta P^{N} =$

$$= \frac{n+2-2(n+1)+n}{1} \Delta x \frac{dP}{dx}$$
$$+ \frac{(n+2)^2-2(n+1)^2+n^2}{1.2} \Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+2)^3-2(n+1)^3+n^3}{1.2.3} \Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+2)^4-2(n+1)^4+n^4}{1.2\ldots4} \Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+2)^5-2(n+1)^5+n^5}{1.2\ldots5} \Delta x^5 \frac{d^5P}{dx^5}$$
$$+ - - - - -$$
$$+ - - - - -$$

$\Delta^2 P^{N+1} =$

$$= \frac{n+3-2(n+2)+n+1}{1} \Delta x \frac{dP}{dx}$$
$$+ \frac{(n+3)^2-2(n+2)^2+(n+1)^2}{1.2} \Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+3)^3-2(n+2)^3+(n+1)^3}{1.2.3} \Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+3)^4-2(n+2)^4+(n+1)^4}{1.2\ldots4} \Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+3)^5-2(n+2)^5+(n+1)^5}{1.2\ldots5} \Delta x^5 \frac{d^5P}{dx^5}$$
$$+ - - - - -$$
$$+ - - - - -$$

Hinc $\Delta^3 P^{N} (= \Delta^2 P^{N+1} - \Delta^2 P^{N}) =$

$$= \frac{(n+3)-3(n+2)+3(n+1)-n}{1} \Delta x \frac{dP}{dx}$$
$$+ \frac{(n+3)^2-3(n+2)^2+3(n+1)^2-n^2}{1.2} \Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+3)^3-3(n+2)^3+3(n+1)^3-n^3}{1.2.3} \Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+3)^4-3(n+2)^4+3(n+1)^4-n^4}{1.2\ldots4} \Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+3)^5-3(n+2)^5+3(n+1)^5-n^5}{1.2..5} \Delta x^5 \frac{d^5P}{dx^5}$$
$$+ - - - - - - -$$
$$+ - - - - - - -$$

$\Delta^3 P^{N+1} =$

$$= \frac{n+4-3(n+3)+3(n+2)-(n+1)}{1} \Delta x \frac{dP}{dx}$$
$$+ \frac{(n+4)^2-3(n+3)^2+3(n+2)^2-(n+1)^2}{1.2} \Delta x^2 \frac{ddP}{dx^2}$$
$$+ \frac{(n+4)^3-3(n+3)^3+3(n+2)^3-(n+1)^3}{1.1.2} \Delta x^3 \frac{d^3P}{dx^3}$$
$$+ \frac{(n+4)^4-3(n+3)^4+3(n+2)^4-(n+1)^4}{1.2\ldots4} \Delta x^4 \frac{d^4P}{dx^4}$$
$$+ \frac{(n+4)^5-3(n+3)^5+3(n+2)^5-(n+1)^5}{1.2\ldots4} \Delta x^5 \frac{d^5P}{dx^5}$$
$$+ - - - - - - -$$
$$+ - - - - - - -$$

Hinc

Hinc $\Delta^4 p\text{N} (=\Delta^3 p\text{N}+1-\Delta^3 p\text{N})$

$$= \frac{n+4-4(n+3)+6(n+2)-4(n+1)+n}{1}\Delta x\frac{dP}{dx}$$

$$+\frac{(n+4)^2-4(n+3)^2+6(n+2)^2-4(n+1)^2+n^2}{1.2}dx^2\frac{ddP}{dx^2}$$

$$+\frac{(n+4)^3-4(n+3)^3+6(n+2)^3-4(n+1)^3+n^3}{1.2.3}dx^3\frac{d^3P}{dx^3}$$

$$+\frac{(n+4)^4-4(n+3)^4+6(n+2)^4-4(n+1)^4+n^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4}$$

$$+\frac{(n+4)^5-4(n+3)^5+6(n+2)^5-4(n+1)^5+n^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5}$$

$$+ - - - - - - - - -$$

$$+ - - - - - - - - -$$

Generatim

$$\Delta^m p\text{N} = \frac{n+m-\frac{m}{1}(n+m-1)+\frac{m}{1}.\frac{m-1}{2}(n+m-2)-\frac{m}{1}..\frac{m-2}{3}(n+m-3)\ldots\pm\frac{m}{1}(n+1)\mp n}{1}\Delta x\frac{dP}{dx}$$

$$+\frac{(n+m)^2-\frac{m}{1}(n+m-1)^2+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^2-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^2\ldots\pm\frac{m}{1}(n+1)^2\mp n^2}{1.2}\Delta x^2\frac{ddP}{dx^2}$$

$$+\frac{(n+m)^3-\frac{m}{1}(n+m-1)^3+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^3-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^3\ldots\pm\frac{m}{1}(n+1)^3\mp n^3}{1.2.3}\Delta x^3\frac{d^3P}{dx^3}$$

$$+\frac{(n+m)^4-\frac{m}{1}(n+m-1)^4+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^4-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^4\ldots\pm\frac{m}{1}(n+1)^4\mp n^4}{1.2\ldots4}\Delta x^4\frac{d^4P}{dx^4}$$

$$+\frac{(n+m)^5-\frac{m}{1}(n+m-1)^5+\frac{m}{1}.\frac{m-1}{2}(n+m-2)^5-\frac{m}{1}..\frac{m-2}{3}(n+m-3)^5\ldots\pm\frac{m}{1}(n+1)^5\mp n^5}{1.2\ldots5}\Delta x^5\frac{d^5P}{dx^5}$$

$$+ - - - - - - - - - -$$

$$+ - - - - - - - - - -$$

Nomi-

Nominatim igitur

$$
\begin{aligned}
\Delta^m P = {} & \frac{m - \frac{m}{1}(m-1) + \frac{m}{1}\cdot\frac{m-1}{1}(m-2) - \frac{m}{1}\ldots\frac{m-2}{1}(m-3)\ \ldots\ \pm\frac{m}{1}1}{1}\,\Delta x\frac{\mathrm{d}P}{\mathrm{d}x} \\
& + \frac{m^2 - \frac{m}{1}(m-1)^2 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^2 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^2 \ldots \pm\frac{m}{1}1^2}{1.2}\,\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2} \\
& + \frac{m^3 - \frac{m}{1}(m-1)^3 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^3 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^3 \ldots \pm\frac{m}{1}1^3}{1.2.3}\,\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3} \\
& + \frac{m^4 - \frac{m}{1}(m-1)^4 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^4 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^4 \ldots \pm\frac{m}{1}1^4}{1.2.3.4}\,\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4} \\
& + \frac{m^5 - \frac{m}{1}(m-1)^5 + \frac{m}{1}\cdot\frac{m-1}{2}(m-2)^5 - \frac{m}{1}\ldots\frac{m-2}{3}(m-3)^5 \ldots \pm\frac{m}{1}1^5}{1.2.\ldots 5}\,\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5} \\
& + \text{- - - - - -} \\
& + \text{- - - - - -} \\
= {} & \frac{\Delta^m . m}{1}\,\Delta x\frac{\mathrm{d}P}{\mathrm{d}x} \\
& + \frac{\Delta^m . m^2}{1.2}\Delta x^2\frac{\mathrm{dd}P}{\mathrm{d}x^2} \\
& + \frac{\Delta^m . m^3}{1.2.3}\Delta x^3\frac{\mathrm{d}^3P}{\mathrm{d}x^3} \\
& + \frac{\Delta^m . m^4}{1.2.3.4}\Delta x^4\frac{\mathrm{d}^4P}{\mathrm{d}x^4} \\
& + \frac{\Delta^m . m^5}{1.2\ldots 5}\Delta x^5\frac{\mathrm{d}^5P}{\mathrm{d}x^5} \\
& + \text{- - - -} \\
& + \text{- - - -}
\end{aligned}
$$

Atqui

Atqui (§. *q. Introd.*) $\Delta^m . m^m = 1.2.3. \ldots m$, & differentiæ ulteriorum ordinum evaneſcunt.

$$\text{Ergo}\quad \Delta^m P = \frac{\Delta^m . m^m}{1.2. \ldots m} \Delta x^m \frac{d^m P}{dx^m}$$

$$+ \frac{\Delta^m . m^{m+1}}{1.2. \ldots (m+1)} \Delta x^{m+1} \frac{d^{m+1} P}{dx^{m+1}}$$

$$+ \frac{\Delta^m . m^{m+2}}{1.2. \ldots (m+2)} \Delta x^{m+2} \frac{d^{m+2} P}{dx^{m+2}}$$

$$+ \frac{\Delta^m . m^{m+3}}{1.2. \ldots (m+3)} \Delta x^{m+3} \frac{d^{m+3} P}{dx^{m+3}}$$

$$+ \frac{\Delta^m . m^{m+4}}{1.2. \ldots (m+4)} \Delta x^{m+4} \frac{d^{m+4} P}{dx^{m+4}}$$

$$+ \quad - \quad - \quad - \quad -$$

$$+ \quad - \quad - \quad - \quad -$$

Itaque differentiæ omnium ordinum functionis cujuslibet alicujus quantitatis mutabilis reducuntur ad differentias poteſtatum numerorum naturalium.

CAPUT QUARTUM

De ſerie Bernoulliana.

§. 35.

Sit $\frac{dZ}{dx} = y$ ratio differentialis data inter quantitates mutabiles x, y, Z. Relatio earum integralis exprimi poteſt per exponentes differentiales omnium ordinum variabilium x & y, quod ſic inveſtigatur.

$$\frac{dZ}{dx} = y$$

$$= y + x\frac{dy}{dx}$$

$$-\left(x\frac{dy}{dx} + \tfrac{1}{2}xx\frac{ddy}{dx^2}\right)$$

$$+\left(\tfrac{1}{2}xx\frac{ddy}{dx^2} + \frac{1}{1.2.3}x^3\frac{d^3y}{dx^3}\right)$$

$$-\left(\frac{1}{1.2.3}x^3\frac{d^3y}{dx^3} + \frac{1}{1.2\ldots4}x^4\frac{d^4y}{dx^4}\right)$$

$$+\left(\frac{1}{1.2\ldots4}x^4\frac{d^4y}{dx^4} + \frac{1}{1.2\ldots5}x^5\frac{d^5y}{dx^5}\right)$$

$$- \quad - \quad - \quad -$$

$$- \quad - \quad - \quad -$$

Itaque

$$Z = C + xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2\ldots4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2\ldots5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2\ldots6}\frac{d^5y}{dx^5} + \ldots$$

Series hæc notatu admodum digna, & cujus fœcundæ applicationes in ſequentibus occurrent, *Bernoulliana* dicitur; quod Joh. Bernoulli primus eam ſub forma hac ſimpliciſſima tradidit. (Vide *ipſius Opera*, T. I. p. 125 ſeqq.)

Exempla. Sit $y = x^m$.

$$\frac{dy}{dx} = mx^{m-1}$$

$$\frac{ddy}{dx^2} = m.m-1x^{m-2}$$

$$\frac{d^3y}{dx^3} = m..m-2x^{m-3}$$

$$\frac{d^4y}{dx^4} = m\ldots m-3x^{m-4}$$

$$\frac{d^5y}{dx^5} = m\ldots.m-4x^{m-5}$$

$$\frac{d^6y}{dx^6} = m\ldots..m-5x^{m-6}$$

$$- \quad - \quad - \quad -$$

$$- \quad - \quad - \quad -$$

Hinc poſito $\frac{dZ}{dx} = x^m$

$$Z = C + x^{m+1}\left(1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1.2..4} + \frac{m...m-3}{1.2...5} - \frac{m...m-4}{1.2...6} + \frac{m...m-5}{1.2...7} - ...\right)$$

Atqui poſito $\frac{dZ}{dx} = x^m$ eſt $Z = C + \frac{1}{m+1}x^{m+1}$ (§. 27.)

Ergo $\frac{1}{m+1} = 1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1.2..4} + \frac{m...m-3}{1.2...5} - \frac{m...m-4}{1.2...6} + \frac{m...m-5}{1.2...7} - .....$

Obſervatio prima. Caſu hoc facile eſt expreſſionum præcedentium identitatem demonſtrare.

Etenim $1 - \frac{m}{1.2} + \frac{m.m-1}{1.2.3} - \frac{m..m-2}{1...4} + \frac{m...m-3}{1...5} - \frac{m...m-4}{1...6} + ...$

$$= \frac{1}{m+1}\left(\frac{m+1}{1} - \frac{m+1.m}{1.2} + \frac{m+1}{1}..\frac{m-1}{3} - \frac{m+1}{1}..\frac{m-2}{4} + \frac{m+1}{1}..\frac{m-3}{5} - \frac{m+1}{1}..\frac{m-4}{6} + ..\right)$$

$$= \frac{1}{m+1}\left(1 - (1-1)^{m+1}\right) = \frac{1}{m+1}.$$

Obſervatio ſecunda. Series Bernoulliana varias induere poteſt formas, prouti quantitas x augetur aut minuitur aliqua quantitate conſtanti a: fit enim eodem omnino modo

$$Z = C + (x+a)y - \frac{(x+a)^2}{1.2}\frac{dy}{dx} + \frac{(x+a)^3}{1.2.3}\frac{ddy}{dx^2} - \frac{(x+a)^4}{1...4}\frac{d^3y}{dx^3} + \frac{(x+a)^5}{1...5}\frac{d^4y}{dx^4} - ....$$

Obſervatio tertia. Cum ſeries Bernoulliana relationem integralem ſub forma omnium ſimpliciſſima raro exhibeat (quod ipſo exemplo præcedente $\frac{dZ}{dx} = x^m$ fit ſatis manifeſtum); ad eam non eſt recurrendum, niſi prius tentatis aliis integrationem propoſitam perficiendi methodis. Seriei hujus collatio cum expreſſionibus, quæ aliis methodis obtinentur, ad varia ducit theoremata, quibus evolvendis non immorabor.

§. 36. In æquatione differentiali $\frac{dZ}{dx} = y$ exponens propoſitus y poteſt eſſe quantitas quælibet ſimplex aut compoſita, dummodo exponentes differentiales omnium ordinum duarum quantitatum variabilium y & x exhiberi poſſint. Hoc monere eo magis e re eſſe cenſui, quod mathematici etiam in calculis ſuperioribus verſatiſſimi non videntur illud ſatis agnoviſſe; uti ex dubiis patet,

patet, quæ ſagaciſſimus HOLLAND amico ſuo clariſs. LAMBERT (*Briefwechſel I. Band* pag. 105.) propoſuit his verbis: „*Ich habe beſonders durch die Bernoulliſche Reihe finden wollen, ob ſich nicht könnte* $\int.y^m dx$ *aus* $\int y\,dx$, *oder welches ich beſonders wünſchte*, $\int.xy\,dx$ *aus* $\int.y\,dx$ *finden laſſen. Mein Bemühen war fruchtlos, wie es noch alle meine Unterſuchungen in dieſem Punct geweſen ſind. Wenn dieſe Vergleichung allgemein möglich iſt (woran ich noch zweifle); ſo ſehe ich die Entdeckung derſelben für eine der wichtigſten an, die man zur Erweiterung der Integral-Rechnung machen kann.*„ (a)

Sit ideo conformiter voto celeb. HOLLAND $\frac{dZ}{dx} = xy$; & ſit $P = xy$.

$$\frac{dP}{dx} = y + x\frac{dy}{dx}$$

$$\frac{ddP}{dx^2} = 2\frac{dy}{dx} + x\frac{ddy}{dx^2}$$

$$\frac{d^3P}{dx^3} = 3\frac{ddy}{dx^2} + x\frac{d^3y}{dx^3}$$

$$\frac{d^4P}{dx^4} = 4\frac{d^3y}{dx^3} + x\frac{d^4y}{dx^4}$$

$$\frac{d^5P}{dx^5} = 5\frac{d^4y}{dx^4} + x\frac{d^5y}{dx^5}$$

$$\frac{d^6P}{dx^6} = 6\frac{d^5y}{dx^5} + x\frac{d^6y}{dx^6}$$

- - - -

- - - -

 Hinc

(a) Hoc eſt: „*Quærebam inprimis, anne ope ſeriei Bernoullianæ ex* $\int.y\,dx$ *liceret* $\int y^m dx$ *aut (quod præcipue optabam)* $\int.xy\,dx$ *eruere. Conatus mei fuerunt irriti, uti hactenus omnes meæ hanc in rem inveſtigationes. Quod ſi hæc comparatio generatim poſſibilis ſit (de quo adhuc dubito); inventio ejus mihi una ex potiſſimis eſſe videtur, quibus calculus integralis perfici poſſit.*

Hinc $Z=C+xxy$

$$-\frac{x^2}{1.2}\left(y+x\frac{dy}{dx}\right]$$

$$+\frac{x^3}{1.2.3}\left(2\frac{dy}{dx}+x\frac{ddy}{dx^2}\right]$$

$$-\frac{x^4}{1.2..4}\left(3\frac{ddy}{dx^2}+x\frac{d^3y}{dx^3}\right]$$

$$+\frac{x^5}{1.2..5}\left(4\frac{d^3y}{dx^3}+x\frac{d^4y}{dx^4}\right]$$

$$-\frac{x^6}{1.2..6}\left[\frac{d^4y}{dx^4}+x\frac{d^5y}{dx^5}\right]$$

$$+\frac{x^7}{1.2..7}\left[\frac{d^5y}{dx^5}+x\frac{d^6y}{dx^6}\right]$$

$$- \; - \; - \; - \; - \; -$$

$$+ \; - \; - \; - \; - \; -$$

$$=C+\frac{1}{1.2}xxy-\frac{x^3}{1.2.3}\frac{dy}{dx}+\frac{x^4}{1.2..4}\frac{ddy}{dx^2}-\frac{x^5}{1.2..5}\frac{d^3y}{dx^3}+\frac{x^6}{1.2..6}\frac{d^4y}{dx^4}-\frac{x^7}{1.2..7}\frac{d^5y}{dx^5}+\ldots\ldots$$

Sit $P=y^m$

$$\frac{dP}{dx}=my^{m-1}\frac{dy}{dx}$$

$$\frac{ddP}{dx^2}=m.m-1y^{m-2}\left[\frac{dy}{dx}\right]^2+my^{m-1}\frac{ddy}{dx^2}$$

$$\frac{d^3P}{dx^3}=m...m-2y^{m-3}\left[\frac{dy}{dx}\right]^3+3m.m-1y^{m-2}\frac{dy}{dx}\cdot\frac{ddy}{dx^2}+my^{m-1}\frac{d^3y}{dx^3}$$

$$\frac{d^4P}{dx^4}=m...m-3y^{m-4}\left[\frac{dy}{dx}\right]^4+6m...m-2y^{m-3}\left[\frac{dy}{dx}\right]^2\frac{ddy}{dx^2}+\begin{matrix}3m.m-1y^{m-2}\left[\frac{ddy}{dx^2}\right]^2\\ 4m.m-1y^{m-2}\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}\end{matrix}+my^{m-1}\frac{d^4y}{dx^4}$$

$$\frac{d^5P}{dx^5}=m...m-4y^{m-5}\left[\frac{dy}{dx}\right]^5+10m...m-3y^{m-4}\left[\frac{dy}{dx}\right]^3\frac{ddy}{dx^2}+\begin{matrix}15m...m-2y^{m-3}\frac{dy}{dx}\left[\frac{ddy}{dx^2}\right]^2\\ 10m...m-2y^{m-3}\left[\frac{dy}{dx}\right]^2\frac{d^3y}{dx^3}\end{matrix}+$$

$$+\begin{matrix}10m.m-1y^{m-2}\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3}\\ 5m.m-1y^{m-2}\frac{dy}{dx}\frac{d^4y}{dx^4}\end{matrix}+m.y^{m-1}\frac{d^5y}{dx^5}$$

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Quibus

Quibus ſubſtitutionibus factis obtinetur valor exponentis integralis per exponentes differentiales omnium ordinum quantitatum y & x.

Sit iterum $\frac{dZ}{dx} = vy$.

Sit $P = vy$

$$\frac{dP}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}$$

$$\frac{ddP}{dx^2} = v\frac{ddy}{dx^2} + 2\frac{dy}{dx}\frac{dv}{dx} + y\frac{ddv}{dx^2}$$

$$\frac{d^3P}{dx^3} = v\frac{d^3y}{dx^3} + 3\frac{dv}{dx}\frac{ddy}{dx^2} + 3\frac{ddv}{dx^2}\frac{dy}{dx} + y\frac{d^3v}{dx^3}$$

$$\frac{d^4P}{dx^4} = v\frac{d^4y}{dx^4} + 4\frac{dv}{dx}\frac{d^3y}{dx^3} + 6\frac{ddv}{dx^2}\frac{ddy}{dx^2} + 4\frac{d^3v}{dx^3}\frac{dy}{dx} + y\frac{d^4v}{dx^4}$$

$$\frac{d^5P}{dx^5} = v\frac{d^5y}{dx^5} + 5\frac{dv}{dx}\frac{d^4y}{dx^4} + 10\frac{ddv}{dx^2}\frac{d^3y}{dx^3} + 10\frac{d^3v}{dx^3}\frac{ddy}{dx^2} + 5\frac{d^4v}{dx^4}\frac{dy}{dx} + y\frac{d^5v}{dx^5}$$

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Hinc $Z = C + vxy$

$$- \frac{x^2}{1.\,2}\left[v\frac{dy}{dx} + y\frac{dv}{dx}\right]$$

$$+ \frac{x^3}{1.\,2.\,3}\left[v\frac{ddy}{dx^2} + 2\frac{dv}{dx}\frac{dy}{dx} + y\frac{ddv}{dx^2}\right]$$

$$- \frac{x^4}{1.2\ldots4}\left[v\frac{d^3y}{dx^3} + 3\frac{dv}{dx}\frac{ddy}{dx^2} + 3\frac{dy}{dx}\frac{ddv}{dx^2} + y\frac{d^3v}{dx^3}\right]$$

$$+ \frac{x^5}{1.2\ldots5}\left[v\frac{d^4y}{dx^4} + 4\frac{dv}{dx}\frac{d^3y}{dx^3} + 6\frac{ddv}{dx^2}\frac{ddy}{dx^2} + 4\frac{d^3v}{dx^3}\frac{dy}{dx} + y\frac{d^4v}{dx^4}\right]$$

$$- \frac{x^6}{1.2\ldots6}\left[v\frac{d^5y}{dx^5} + 5\frac{dv}{dx}\frac{d^4y}{dx^4} + 10\frac{ddv}{dx^2}\frac{d^3y}{dx^3} + 10\frac{d^3v}{dx^3}\frac{ddy}{dx^2} + 5\frac{d^4v}{dx^4}\frac{dy}{dx} + y\frac{d^5v}{dx^5}\right]$$

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§. 37. Quemadmodum relatio differentialis primi ordinis $\frac{dZ}{dx} = y$ ducit ad relationem integralem, qua Z per x & y determinatur: ita etiam relatio differentialis incomplexa $\frac{d^m Z}{dx^m} = y$ ducit ad relationem integralem inter $\frac{d^{m-1}Z}{dx^{m-1}}$ & easdem variabiles; ac deinde ad determinationem aliorum exponentium differentialium inferiorum ipſarum Z & x.

1°. Sit $\frac{ddZ}{dx^2} = y$.

Erit (§. 35.)

$$\frac{dZ}{dx} = C + xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2...5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2...6}\frac{d^5y}{dx^5} + \ldots$$

Sit $\frac{dZ}{dx} = C + P$.

$$\text{Itaque } \frac{dP}{dx} = y$$

$$\frac{ddP}{dx^2} = \frac{dy}{dx}$$

$$\frac{d^3P}{dx^3} = \frac{ddy}{dx^2}$$

$$\frac{d^4P}{dx^4} = \frac{d^3y}{dx^3}$$

$$\frac{d^5P}{dx^5} = \frac{d^4y}{dx^4}$$

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$$\text{Hinc } Z = C' + Cx + xxy - \frac{x^3}{1.2}\frac{dy}{dx} + \frac{x^4}{1.2.3}\frac{ddy}{dx^2} - \frac{x^5}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^6}{1.2...5}\frac{d^4y}{dx^4} - \ldots$$

$$- \frac{x^2}{1.2}y + \frac{x^3}{1.2.3}\frac{dy}{dx} - \frac{x^4}{1.2...4}\frac{ddy}{dx^2} + \frac{x^5}{1.2...5}\frac{d^3y}{dx^3} - \frac{x^6}{1.2...6}\frac{d^4y}{dx^4} + \ldots$$

$$= C' + Cx + \frac{x^2}{1.2}y - \frac{2x^3}{1.2.3}\frac{dy}{dx} + \frac{3x^4}{1.2...4}\frac{ddy}{dx^2} - \frac{4x^5}{1.2...5}\frac{d^3y}{dx^3} + \frac{5x^6}{1.2...6}\frac{d^4y}{dx^4} - \ldots$$

Sit $\frac{d^3 Z}{dx^3} = y$.

Hinc $\frac{dZ}{dx} = C' + Cx + \frac{x^2}{1.2}y - \frac{2x^3}{1.2.3}\frac{dy}{dx} + \frac{3x^4}{1.2...4}\frac{ddy}{dx^2} - \frac{4x^5}{1.2...5}\frac{d^3y}{dx^3} + \frac{5x^6}{1.2...6}\frac{d^4y}{dx^4} - \ldots$
$= C' + Cx + P.$

$$\frac{dP}{dx} = xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1.2...4}\frac{d^3y}{dx^3} + \frac{x^5}{1.2...5}\frac{d^4y}{dx^4} - \frac{x^6}{1.2...6}\frac{d^5y}{dx^5} + \ldots$$

$$\frac{ddP}{dx^2} = y$$

$$\frac{d^3P}{dx^3} = \frac{dy}{dx}$$

$$\frac{d^4P}{dx^4} = \frac{ddy}{dx^2}$$

$$\frac{d^5P}{dx^5} = \frac{d^3y}{dx^3}$$

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Proinde $Z = C'' + C'x + \frac{1}{2}Cx^2$

$$+ \frac{x^3}{1.2.3}y - \frac{3x^4}{1.2..4}\frac{dy}{dx} + \frac{6x^5}{1.2...5}\frac{ddy}{dx^2} - \frac{10x^6}{1.2...6}\frac{d^3y}{dx^3} + \frac{15x^7}{1.2...7}\frac{d^4y}{dx^4} - \frac{21x^8}{1.2...8}\frac{d^5y}{dx^5} + \ldots.$$

Hinc porro, ſi $\frac{d^4Z}{dx^4} = y$

erit $Z = C''' + C''x + \frac{1}{2}C'x^2 + \frac{1}{1.2.3}Cx^3$

$$+ \frac{x^4}{1.2...4}y - \frac{4x^5}{1.2...5}\frac{dy}{dx} + \frac{10x^6}{1.2...6}\frac{ddy}{dx^2} - \frac{20x^7}{1.2...7}\frac{d^3y}{dx^3} + \frac{35x^8}{1.2...8}\frac{d^4y}{dx^4} - \frac{56x^9}{1.2...9}\frac{d^5y}{dx^5} + \ldots.$$

Generatim ſit $\frac{d^mZ}{dx^m} = y$.

Fiet $Z = C^{\mathrm{M-I}} + C^{\mathrm{M-II}}x + \frac{1}{1.2}C^{\mathrm{M-III}}x^2 + \frac{1}{1.2.3}C^{\mathrm{M-IV}}x^3 + \ldots \frac{1}{1.2...m-2}C'x^{m-2} + \frac{1}{1.2...m-1}Cx^{m-1}$

$$+ \frac{x^m}{1.2...m}y - \frac{m.x^{m+1}}{1.2...m+1}\frac{dy}{dx} + \frac{\frac{m}{1}.\frac{m+1}{2}x^{m+2}}{1.2...m+2}\frac{ddy}{dx^2} - \frac{\frac{m}{1}\ldots\frac{m+2}{3}x^{m+3}}{1.2...m+3}\frac{d^3y}{dx^3} +$$

$$+ \frac{\frac{m}{1}\ldots\frac{m+3}{4}x^{m+4}}{1.2...m+4}\frac{d^4y}{dx^4} - \frac{\frac{m}{1}\ldots\frac{m+4}{5}x^{m+5}}{1.2...m+5}\frac{d^5y}{dx^5} + \ldots..$$

Obſervatio. Ex his formulis plurima curioſa ſaltem theoremata deduci poſſent; conferendo eas cum formulis, quæ cum ipſis identicæ eſſe debent, & quæ aut integratione immediata, aut methodis a priore diverſis obtinentur. Et hic repetendum eſt (a fortiori) monitum, ut ad ſeries Bernoullianas non prius recurratur, quam alia ſubſidia fuerint exhauſta.

Exemplum.

Sit $\frac{d^m Z}{dx^m} = x^n.$

$$\frac{d^{m-1}Z}{dx^{m-1}} = C + \frac{1}{n+1}x^{n+1}$$

$$\frac{d^{m-2}Z}{dx^{m-2}} = C' + Cx + \frac{1}{n+1\,.\,n+2}x^{n+2}$$

$$\frac{d^{m-3}Z}{dx^{m-3}} = C'' + C'x + \frac{1}{1.2}Cx^2 + \frac{1}{n+1\ldots n+3}x^{n+3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$Z = C^{M-I} + C^{M-II}x + \frac{1}{1.2}C^{M-III}x^2 + \ldots + \frac{1}{1.2..m-1}Cx^{m-1} + \frac{1}{n+1\ldots n+m}x^{n+m}.$$

Caput Quintum.

De Tangentibus.

§. 38.

Definitio. Sit arcus curvæ, isque totus aut concavus aut convexus verſus diametrum, ad quam refertur: per punctum aliquod hujus arcus ducatur linea recta, quæ ipſi ita occurrat, ut ad utramque puncti hujus partem extra curvam jaceat. Hæc recta dicitur *tangens* curvæ, & punctum, ubi curvæ occurrit, dicitur *punctum contactus.*

Per punctum contactus ducatur recta diametro ordinatim applicata. Si tangens occurrat diametro, ad quam curva refertur: pars diametri ordinatam inter a puncto contactus ductam & punctum, ubi tangens occurrit diametro, comprehenſa, dicitur *ſubtangens.*

Sci-

Scilicet curva AMM' referatur ad diametrum AP per rectas ipſi ordinatim applicatas MP, $M'P'$. Per punctum M arcus MM', qui ex utraque puncti M parte ad diametrum ſit concavus aut convexus, ducatur recta MT, quæ curvæ in puncto M ita occurrat, ut ad utramque puncti M partem extra curvam cadat: hæc recta MT dicitur tangens curvæ, & punctum M dicitur punctum contactus. Sit MP diametro ordinatim applicata, & tangens occurrat diametro in T: linea PT dicitur ſubtangens. Fig. 11.

Quæcunque dicentur de arcubus verſus diametrum concavis, facile (mutatis mutandis) ad arcus verſus diametrum convexos applicantur: quare (brevitatis cauſa) de prioribus tantum dicere ſufficiat.

Obſervatio. Ex puncto M ad aliud quodpiam punctum M' arcus AMM' ducatur chorda MM', quæ producta diametro occurrat in S. Per punctum M' ducantur recta $M'P'$ diametro ordinatim applicata; & recta $M'Q$ diametro parallela, quæ ipſi MP occurrat in Q. Linea PS dicitur *ſubſecans.*

Fit ſemper $MP : PS = MQ : M'Q$; hoc eſt: ratio mutationum ſimultanearum ordinatæ & abſciſſæ diametri æquatur rationi ordinatæ ad ſubſecantem. Prior itaque ratio major eſt aut minor ratione ordinatæ ad ſubtangentem, prouti ſubſecans minor eſt aut major ſubtangente: nempe in caſu figuræ, ubi arcus MM' concavus eſt verſus diametrum, prouti punctum M' ſitum eſt inter puncta M & S, aut ſecus. Quoniam vero puncto M' propius propiusque ad punctum M accedente (ita ut mutationes ſimultaneæ ordinatæ & abſciſſæ diametri continue fiant minores), punctum S continue accedit ad punctum T; ita ut ſubſecans contiuue minus differat a ſubtangente: proclive eſt ſuſpicari, rationem ordinatæ ad ſubtangentem poſſe ex ratione differentiali (ſi qua fuerit) ordinatæ ad ſubſecantem deduci; quod theoremate ſequente ſtabilitur.

§. 39. *Theorema.* Ex puncto aliquo cujuslibet arcus, qui totus ſit verſus diametrum, ad quam refertur, concavus (aut convexus), ducatur recta, quæ arcum tangat & diametro occurrat. Dico: rationem mutationum ſimultanearum rectæ diametro ab eodem puncto ordinatim applicatæ & abſciſſæ diametri fieri poſſe minorem quacunque ratione data, quæ major ſit ratione data ordi-

natæ

natæ ad ſubtangentem; majorem vero quacunque ratione data, quæ minor ſit prædicta ratione.

Sit MP diametro AP ordinatim applicata; & tangens MT, quæ diametro occurrat in T. Detur ratio $MP : PS$ $\substack{\text{major}\\ \text{minor}}$ ratione $MP : PT$; ideoque ſit PS $\substack{\text{minor}\\ \text{major}}$ ipſa PT. Dico: rationem mutationum ſimultanearum ordinatæ MP & abſciſſæ diametri AP poſſe fieri $\substack{\text{minorem}\\ \text{majorem}}$ ratione propoſita $MP : PS$.

Arcus propoſiti ſumatur punctum quodcunque m, priore quidem caſu ſitum ad eaſdem ordinatæ MP partes, ad quas eſt punctum T; poſteriore ad partes oppoſitas. Jungatur recta Mm; quæ, cum arcus (hyp.) ſit verſus diametrum concavus, tota intra eum cadet. Per punctum m ducantur rectæ mp ordinatim diametro applicata; atque mq diametro parallela, quæ ordinatæ MP ipſi vel productæ in puncto q occurrat.

Fig. 11. 1°. Erit ratio $Mq : mq$ vel $\left\{\substack{\text{minor priore caſu}\\ \text{major poſteriori caſu}}\right\}$ quam data ratio $MP : PS$; & tunc effectum erit, quod proponitur.

Vel ratio $Mq : mq$ erit rationi datæ $MP : PS$ æqualis, aut ipſa $\left\{\substack{\text{major priore caſu}\\ \text{minor caſu poſteriore}}\right\}$.

Si prius: ob $Mq : mq = MP : PS$ recta producta Mm incidet in punctum S.

Si poſterius: recta Mm producta diametrum in puncto s ita ſecabit, ut
2°. & 3°. punctum S cadat inter puncta s & T, ob $MP : Ps (= Mq : mq) \gtrless MP : PS$; proinde $Ps \lessgtr PS$. Tum igitur recta SM, utpote intra angulum TMs, eique ad verticem oppoſitum tMm ſita, arcum Mm verſus diametrum (hyp.) concavum in puncto aliquo M' citra vel ultra M ſito ita ſecabit, ut ſegmentum ipſius MM' intra arcum jaceat. Per M' agantur recta $M'P'$ diametro ordinatim applicata, atque $M'Q$ diametro parallela, quæ ordinatæ MP ipſi vel productæ in puncto Q occurrat. Porro ſumatur punctum quodcunque arcus Mm quidem, ſi producta Mm in punctum S incidit; arcus autem MM', ſi ſecus: & per hoc punctum N agatur diametro parallela NR, quæ ordinatæ MP in puncto R, chordæ vero Mm vel MM' in puncto N' occurrat.

Erit ſemper $MR : N'R = MP : PS$.

Atqui

Atqui $NR > N'R$, ſi puncta m, M', N ad easdem ordinatæ MP partes jacent, ad quas eſt punctum T; tum ergo $MR : NR < MR : N'R < MP : PS$.

Et $NR < N'R$, ſi puncta m, M', N, atque punctum T jacent ad partes ordinatæ MP oppoſitas; tum itaque $MR : NR > MR : N'R > MP : PS$.

Sed ratio $MR : NR$ eſt ratio mutationum ſimultanearum ordinatæ MP & abſciſſæ diametri AP. Ergo ratio harum mutationum ſimultanearum poteſt reddi $\begin{matrix}\text{minor}\\\text{major}\end{matrix}$ quacunque ratione propoſita, quæ $\begin{matrix}\text{major}\\\text{minor}\end{matrix}$ fuerit ratione ordinatæ ad ſubtangentem.

§. 40. *Corollaria.* 1°. Ratio ordinatæ ad ſubtangentem eſt limes rationis mutationum ſimultanearum rectæ a puncto contactus ad diametrum ordinatim applicatæ, & lineæ ex diametro abſciſſæ; ſeu prior ratio eſt ratio differentialis harum linearum.

Scilicet ſit $MP = y$; $AP = x$; $\frac{MP}{PT} = \frac{dy}{dx}$; ſeu $PT = y\frac{dx}{dy}$.

Ex constructione liquet: ſubtangentem eſſe limitem (ſi quis detur) ſubſecantis; quod etiam evinci poteſt, ut ſequitur.

Sit $MP = y$; $M'P' = y'$; $PP' = \Delta x$.

Priore caſu eſt $M'P' = y - \Delta x . \frac{dy}{dx} + \frac{\Delta x^2}{1.2} . \frac{ddy}{dx^2} - \frac{\Delta x^3}{1.2.3} . \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2...4} . \frac{d^4y}{dx^4} - \ldots\ldots$

(§. 33.)

Altero caſu $M'P' = y + \Delta x . \frac{dy}{dx} + \frac{\Delta x^2}{1.2} . \frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3} . \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2...4} . \frac{d^4y}{dx^4} + \ldots\ldots$

Hinc $MQ = \Delta x . \frac{dy}{dx} \mp \frac{\Delta x^2}{1.2} . \frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3} . \frac{d^3y}{dx^3} \mp \frac{\Delta x^4}{1.2...4} . \frac{d^4y}{dx^4} \dotplus \ldots\ldots$

$$PS = y \times \frac{1}{\frac{dy}{dx} \mp \frac{\Delta x}{1.2} . \frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2.3} . \frac{d^3y}{dx^3} \mp \frac{\Delta x^3}{1.2...4} . \frac{d^4y}{dx^4} + \ldots\ldots}$$

Si quis ſit limes poſteriorioris membri, erit etiam aliquis limes prioris. Sed (§. 16.) limes poſterioris membri (ſi quis ſit) eſt $y\frac{1}{\frac{dy}{dx}} = y\frac{dx}{dy}$. Ergo: limes prioris membri, nempe PT, eſt $y\frac{dx}{dy}$.

2°. Ratio æqualitatis eſt etiam limes rationis tangentis ad ſecantem; quatenus utraque recta hinc puncto contactus, inde punctis, ubi diametro occurrunt, terminatur. Et proinde tangens eſt limes ſecantis.

3°. Li-

3°. Limes rationis ſecantis ad ſubſecantem æqualis eſt rationi tangentis ad ſubtangentem.

§. 41. Hinc quoque determinantur tam angulus, ſub quo tangens occurrit (ſi fieri poſſit) diametro, quam angulus, quem tangens facit cum recta, quæ a puncto contactus ordinatim diametro applicatur.

Nempe ſi angulus P fuerit rectus: erit $\text{tang.} MTP = \frac{MP}{PT} = \frac{dy}{dx}$

$$\& \ \cot. MTP = \text{tang.} PMT = \frac{PT}{MP} = \frac{dx}{dy}$$

Si vero angulus APM non fuerit rectus: erit

$$\cot. MTP = \frac{PT}{MP} \text{coſec.} APM - \cot. APM = \frac{dx}{dy} \text{coſec.} P - \cot. P;$$

$$\& \ \cot. 180° - MTP = \cot. APM - \frac{dx}{dy} \text{coſec.} APM$$

$$\cot. PMT = \frac{MP}{PT} \text{coſec.} APM - \cot. APM = \frac{dy}{dx} \text{coſec.} P - \cot. P;$$

$$\& \ \cot. 180° - PMT = \cot. APM - \frac{dy}{dx} \text{coſec.} APM. \ (a)$$

§. 42. Tranſeo ad exempla, quibus uſus præcedentium formularum fiat familiaris.

Sit æquatio propoſita $y^{m+n} = a^m x^n$ (quæ eſt æquatio parabolarum, m & n exiſtentibus numeris poſitivis).

Erit $(m+n) y^{m+n-1} \frac{dy}{dx} = n a^m x^{n-1}$

$$\frac{dy}{dx} = \frac{n}{m+n} \cdot \frac{a^m x^{n-1}}{y^{m+n-1}} = \frac{n}{m+n} \cdot \frac{y}{x} = \frac{n}{m+n} \left(\frac{a}{x}\right)^{\frac{m}{m+n}} = \frac{n}{m+n} \left(\frac{a}{y}\right)^{\frac{m+n}{n}}$$

$$\frac{dx}{dy} = \frac{m+n}{n} \cdot \frac{x}{y} = \frac{m+n}{n} \left(\frac{x}{a}\right)^{\frac{m}{m+n}} = \frac{m+n}{n} \left(\frac{y}{a}\right)^{\frac{m+n}{n}}$$

$$y \frac{dx}{dy} = PT = \frac{m+n}{n} x.$$

cot.

(a) Formulæ hæ fluunt ex hoc theoremate noto trigonometriæ planæ.
Sint A, B, C . . . latera trianguli rectilinei,
a, b, c . . . anguli his lateribus reſpective oppoſiti.
Datis lateribus A & B, & angulo c, quem continent; reliqui anguli a, b, immediate determinantur, ut ſequitur: $\cot. a = \frac{B}{A} \text{coſec.} c - \cot. c$

$$\cot. b = \frac{A}{B} \text{coſec.} c - \cot. c.$$

$$\cot. MTP = \frac{m+n}{n}\left(\frac{x}{a}\right)^{\frac{m}{m+n}} \operatorname{coſec}. APM - \cot. APM;$$

$$\cot. 180^\circ - MTP = \cot. APM - \frac{m+n}{n} \cdot \frac{x}{a} \cdot \frac{m}{m+n} \operatorname{coſec}. APM$$

$$\cot. PMT = \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{coſec}. APM - \cot. APM;$$

$$\cot. 180^\circ - PMT = \cot. APM - \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{coſec}. APM.$$

Sit $x = 0$; erit $PT = \frac{m+n}{n} \cdot 0$: proinde altero puncto T, quod ductum tangentis determinet, deficiente; ejus poſitio foret indeterminata, niſi in promtu eſſet angulus, ſub quo tangens diametro ad verticem occurrit. Quoniam autem $x = 0$; $\left(\frac{x}{a}\right)^{\frac{m}{m+n}} = 0$; proinde $\cot. 180^\circ - MTP = \cot. APM$; & $180^\circ - MTP = APM$. Ideo recta, quæ parabolas in vertice diametri tangit, parallela eſt rectis, quæ huic diametro ordinatim applicantur.

Etenim ſi fieri poſſet, ut recta, quæ ex vertice diametri alicujus parabolæ ducitur parallela rectis huic diametro ordinatim applicatis, occurreret iterum parabolæ; ſit y pars iſtius rectæ intra parabolam comprehenſa. Eſſet $y^{m+n} = a^m 0^n$; & proinde $y = 0$, contra hyp.

Creſcente x, quantitas $\left(\frac{a}{x}\right)^{\frac{m}{m+n}}$ continue decreſcit, nunquam vero fit $= 0$. Hinc eſt ſemper $\cot. 180^\circ - PMT < \cot. APM$. Proinde angulo APM exiſtente recto aut acuto, eſt ſemper $180^\circ - PMT > APM$; & $PMT + APM < 180^\circ$. Sit autem APM obtuſus, cujus ſupplementum ſit MPA'.

Erit $\cot. PMT = \frac{n}{m+n}\left(\frac{a}{x}\right)^{\frac{m}{m+n}} \operatorname{coſec}. APM + \cot. MPA'$; Fig. 12.

hinc eſt ſemper $\cot. PMT > \cot. MPA'$; $PMT < MPA' < 180^\circ - MPT$;

unde iterum $PMT + MPT < 180^\circ$.

Tangentes ideo cujusvis parabolæ nunquam fiunt diametro parallelæ; ſed ad parallelismum accedunt eo magis, quo remotiora a vertice ſunt puncta curvæ, ad quæ ducuntur.

Exem-

Exemplum 2. Sit $y^m x^n = a^{m+n}$, quæ eſt æquatio hyperbolarum ad aſymptotos relatarum, exiſtentibus m & n numeris poſitivis.

Eſt ideo $my^{m-1}x^n \frac{dy}{dx} + ny^m x^{n-1} = 0$;

$$mx\frac{dy}{dx} \quad + ny \quad = 0;$$

$$\frac{dy}{dx} = -\frac{n}{m}\frac{y}{x} = -\frac{n}{m}.\left(\frac{a}{x}\right)^{\frac{m+n}{m}} = -\frac{n}{m}\left(\frac{y}{a}\right)^{\frac{m}{n}}$$

$$\frac{dx}{dy} = -\frac{m}{n}\frac{x}{y} = -\frac{m}{n}.\left(\frac{x}{a}\right)^{\frac{m+n}{m}} = -\frac{m}{n}\left(\frac{a}{y}\right)^{\frac{m}{n}}$$

$$y\frac{dx}{dy} = -\frac{m}{n}x;\quad PT = -\frac{m}{n}AP.$$

Scilicet ratio ſubtangentis ad abſciſſam diametri eſt etiam ratio data; ſed ſubtangens & abſciſſa diametri ad partes ordinatæ MP oppoſitas jacent.

Fig. 13. $$\text{cot}.MTP = \frac{PT}{MP}\text{coſec}.MPT - \text{cot}.MPT = \frac{m}{n}\frac{AP}{MP}\text{coſec}.MPA + \text{cot}.MPA$$

$$= \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{m}}\text{coſec}.MPA + \text{cot}.MPA$$

$$\text{cot}.PMT = \frac{MP}{PT}\text{coſec}.MPT - \text{cot}.MPT = \frac{n}{m}\left(\frac{a}{x}\right)^{\frac{m+n}{m}}\text{coſec}.MPA + \text{cot}.MPA.$$

Sit MPA rectus aut acutus: cot. MTP ſemper major eſt quam cot. MPA; ſed ad eam eo propius accedit, quo minor x. Sit autem MPA obtuſus;

$$\text{cot}.MTP = \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{n}}\text{coſec}.MPT - \text{cot}.MPT$$

$$\text{cot}.180^\circ - MTP = \text{cot}.MPT - \frac{m}{n}\left(\frac{x}{a}\right)^{\frac{m+n}{n}}\text{coſec}.MPT.$$

Ideo cot. $180^\circ - MTP$ ſemper minor cot. MPT; ſed ab ea differt eo minus, quo minor x. Scilicet tangens MT nunquam fit parallela ordinatis MP, ſed ad parallelismum accedit eo magis, quo minor abſciſſa AP.

Sit MPA rectus aut acutus: cot. PMT ſemper major quam cot. MPA; ab ea differt eo magis, quo major x. Sit MPA obtuſus:

$$\text{cot}.180^\circ - PMT = \text{cot}.MPT - \frac{n}{m}\left(\frac{a}{x}\right)^{\frac{m+n}{n}}\text{coſec}.P.$$

Proinde cot. $180^\circ - PMT$ ſemper minor cot. MPT (unde $MPT + PMT < 180^\circ$); ſed

ſed ab ea differt eo minus, quo major eſt x ſeu AP. Scilicet tangens MT nunquam fit parallela aſymptoto AP; ſed ad parallelismum accedit eo magis, quo major AP.

Exemplum 3. Sit $y^m = \frac{b^m}{a^m}(x^m - a^m)$, quæ eſt etiam æquatio hyperbolarum ad diametros relatarum.

$$\text{Erit } my^{m-1}\frac{dy}{dx} = m\frac{b^m}{a^m}x^{m-1}$$

$$\frac{dy}{dx} = \frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}}$$

$$\frac{dx}{dy} = \frac{a^m}{b^m}\cdot\frac{y^{m-1}}{x^{m-1}}$$

$$y\frac{dx}{dy} = PT = \frac{\frac{a^m}{b^m}\cdot\frac{b^m}{a^m}(x^m - a^m)}{x^{m-1}}$$

$$= \frac{x^m - a^m}{x^{m-1}} = x - \frac{a^m}{x^{m-1}}.$$

$$\text{Hinc } x - PT = \frac{a^m}{x^{m-1}} = a\left(\frac{a}{x}\right)^{m-1}$$

Subtangens igitur PT ſemper eſt minor quam abſciſſa diametri. Quoniam autem ex natura curvæ $x > a$; quo major x, eo minor $\frac{a}{x}$, & a fortiori eo minor $\left(\frac{a}{x}\right)^{m-1}$.

Proinde x poteſt fieri adeo magna, ut exceſſus, quo abſciſſa diametri ſubtangentem ſuperat, minor fiat quacunque quantitate propoſita.

$$\text{cot. } MTP = \frac{a}{b}\left(\frac{x^m - a^m}{x^m}\right)^{\frac{m-1}{m}} \text{coſec. } P - \text{cot. } P$$

$$= \frac{a}{b}\left(1 - \frac{a^m}{x^m}\right)^{\frac{m-1}{m}} \text{coſec. } P - \text{cot. } P.$$

Hinc eſt ſemper $\text{cot. } MTP + \text{cot. } P > \frac{a}{b}\text{coſec. } P$.

Sit $P = 90^\circ$; cot. $MTP > \frac{a}{b}$

tang. $MTP < \frac{b}{a}$: ſed $\frac{b}{a}$ eſt limes tangentis creſcentis anguli MTP.

Et in genere: ſin. $MTP + P > \frac{a}{b}$ ſin. MTP

Sed ſin. PMT : ſin. $MPT = TP : MP$; ergo eſt ſemper $TP > MP \cdot \frac{a}{b}$.

Sit $x = a$; $a - PT = a$; $PT = 0$; unde altero puncto T, quod ductum tangentis determinat, deficiente, ejus poſitio foret indeterminata, niſi in promtu eſſet angulus, ſub quo tangens diametro ad verticem occurrit. Fit autem cot. $MTP = -$ cot. P; cot. $180^\circ - MTP =$ cot. P; $180^\circ - MTP = P$; & proinde tangens ad verticem eſt parallela rectis diametro ordinatim applicatis.

Exemplum 4. Sit $y^m = \frac{b^m}{a^m}(a^m - x^m)$, quæ eſt æquatio ellipſium.

$$my^{m-1}\frac{dy}{dx} = -m\frac{b^m}{a^m}x^{m-1}$$

$$\frac{dy}{dx} = -\frac{b^m}{a^m}\left(\frac{x}{y}\right)^{m-1}$$

$$\frac{dx}{dy} = -\frac{a^m}{b^m}\left(\frac{y}{x}\right)^{m-1}$$

$$y\frac{dx}{dy} = PT = -\frac{a^m}{b^m}\frac{y^m}{x^{m-1}} = -\frac{a^m - x^m}{x^{m-1}}.$$ Signum $-$ indicat ſubtangentem & abſciſſas ſitas eſſe ad partes diverſas lineæ diametro ordinatim applicatæ.

$$\text{cot. } 180^\circ - T = \text{cot. } P + \frac{a}{b}\left(\frac{a^m}{x^m} - 1\right)^{\frac{m-1}{m}} \text{coſec. } P$$

$$\text{cot. } 180^\circ - M = \text{cot. } P + \frac{b}{a}\left(\frac{a^m}{a^m - x^m} - 1\right)^{\frac{m-1}{m}} \text{coſec. } P.$$

Sit $x = a$; $PT = 0$; unde ductus tangentis foret indeterminatus. Sed propter $\frac{a^m}{x^m} - 1 = 0$, fit cot. $180^\circ - T =$ cot. P, ſeu $T + P = 180^\circ$. Unde ad verticem diametri tangens eſt rectis diametro ordinatim applicatis parallela.

Quem-

Quemadmodum $y^m = \frac{b^m}{a^m}(a^m - x^m)$

eſt etiam $x^m = \frac{a^m}{b^m}(b^m - y^m)$.

Unde facto $x = a$, ſeu $y = b$, pariter eſt tangens priori diametro parallela.

Ex præcedentibus facile ſolvuntur (ſi fieri poſſit) problemata ſequentia.

§. 43. *Problema.* Ducere lineam rectæ poſitione datæ parallelam, quæ (ſi fieri poſſit) curvam datam contingat.

Solutio. Ducatur recta quævis, ad quam tanquam diametrum curva referatur per rectas huic diametro ordinatim applicatas. Jam ex natura data curvæ determinetur ejus æquatio reſpectu illius diametri, & inde determinetur ratio differentialis $\frac{dx}{dy}$ vel $\frac{dy}{dx}$. Tum in æquatione $\cot. 180^\circ - T = \cot. p - \frac{dx}{dy} \operatorname{coſec.} p$ datur angulus T, & eliminari poteſt $\frac{dx}{dy}$; hinc determinatio quantitatum x & y ſemper reducitur ad ſolutionem æquationis alicujus datæ; proinde problema tanquam ſolutum haberi poteſt (quantum permittit imperfecta æquationum theoria).

Exemplum 1. Sit parabola, cujus æquatio $y^{m+n} = a^m x^n$, ad axem relata rectis huic perpendiculariter ordinatim applicatis. Et ducenda ſit recta, quæ parabolam hanc tangat, & axi occurrat ſub angulo dato T.

Eſt igitur $\cot. T = \frac{dx}{dy}$.

Sed ex æquatione parabolæ eſt $\frac{dx}{dy} = \frac{m+n}{n} \cdot \frac{y^{m+n-1}}{a^m x^{n-1}} = \frac{m+n}{n} \cdot \frac{x}{y}$.

Hinc $\frac{x}{y} = \frac{n}{m+n} \cot. T$

ſeu $x : y = n \cot. T : m+n$

$x^n : y^n = n^n \cot.^n T : (m+n)^n$ $\quad$ $x^{m+n} : y^{m+n} = n^{m+n} \cot.^{m+n} T : (m+n)^{m+n}$

unde $y^m : a^m = n^n \cot.^n T : (m+n)^n$ $\quad$ unde $x^m : a^m = n^{m+n} \cot.^{m+n} T : (m+n)^{m+n}$

$$y = a\left(\frac{n}{m+n}\right)^{\frac{n}{m}} \cot.^{\frac{n}{m}} T \qquad x = a\left(\frac{n}{m+n}\right)^{\frac{m+n}{m}} \cot.^{\frac{m+n}{m}} T.$$

Exem-

Exemplum 2. Curva propoſita ſit hyperbola, cujus æquatio $y^m = \frac{b^m}{a^m}(x^m - a^m)$; unde $\frac{dx}{dy} = \frac{a^m}{b^m} \cdot \frac{y^{m-1}}{x^{m-1}}$.

Fit ideo $\cot. T = \frac{a^m}{b^m} \cdot \frac{y^{m-1}}{x^{m-1}}$.

$$= \frac{a^m}{b^m} \cdot \frac{y^m}{yx^{m-1}} = \frac{x^m - a^m}{yx^{m-1}}$$

$$y = \frac{x^m - a^m}{x^{m-1}} \tang. T;$$

$$\frac{b}{a} x^{m-1} = (x^m - a^m)^{\frac{m-1}{m}} \tang. T$$

$$\left(\frac{b}{a}\right)^{\frac{m}{m-1}} x^m = (x^m - a^m) \tang.^{\frac{m}{m-1}} T.$$

$$x^m : x^m - a^m = \tang.^{\frac{m}{m-1}} T : \left(\frac{b}{a}\right)^{\frac{m}{m-1}}.$$

Quoniam autem $x^m > x^m - a^m$

debet eſſe $\tang.^{\frac{m}{m-1}} T > \left(\frac{b}{a}\right)^{\frac{m}{m-1}}$

& proinde $\tang. T > \frac{b}{a}$, uti in Ex. 3. §. 42.

§. 44. *Problema.* Ab puncto poſitione dato ducere rectam, quæ (ſi fieri poſſit) curvam poſitione datam contingat.

Solutio. Per punctum datum T ducatur recta quævis, ad quam tanquam diametrum curva data referatur per rectas huic ordinatim applicatas. Ex data curvæ æquatione, quatenus ad hanc diametrum refertur, determinetur ratio differentialis $\frac{dy}{dx}$ vel $\frac{dx}{dy}$, quæ ſubſtituatur in æquatione ſubtangentis $PT = y\frac{dx}{dy}$. Quoniam diſtantia puncti T ab origine abſciſſarum data ſupponitur, æquatio inde nata afficietur tantum quantitatibus incognitis x & y, quæ cum æquatione curvæ combinata ducet ad ſolutionem problematis propoſiti (quousque per theoriam æquationum licet).

Exemplum 1. Curva data ſit ellipſis, cujus æquatio $y^m = \frac{b^m}{a^m}(a^m - x^m)$; & punctum T ſitum ſit ſuper diametro, ad quam curva refertur.

Eſt

Eſt ideo $\frac{dy}{dx} = -\frac{b^m}{a^m} \frac{x^{m-1}}{y^{m-1}}$, quod indicat ſubtangentem & abſciſſam diametri ſitas eſſe ad diverſas partes ordinatæ.

$y\frac{dy}{dx} = PT = \frac{a^m}{b^m} \cdot \frac{y^m}{x^{m-1}} = \frac{a^m - x^m}{x^{m-1}} = a\left(\frac{a}{x}\right)^{m-1} - x$. Diſtantia puncti T ab origine abſciſſarum ſit l: fit $l - x = a\left(\frac{a}{x}\right)^{m-1} - x$; $\left(\frac{x}{a}\right)^{m-1} = \frac{a}{l}$; $x = a\sqrt[m-1]{\frac{a}{l}}$.

Exemplum 2. Curva data ſit hyperbola, cujus æquatio $y^m = \frac{b^m}{a^m}(x^m - a^m)$; & punctum T ſitum ſit ſuper diametro, ad quam curva refertur.

Eſt ut ſupra, $\frac{dy}{dx} = \frac{b^m}{a^m} \cdot \frac{y^m}{x^{m-1}}$.

$$y\frac{dx}{dy} = \frac{a^m}{b^m} \cdot \frac{y^m}{x^{m-1}} = \frac{x^m - a^m}{x^{m-1}} = x - a\left(\frac{a}{x}\right)^{m-1} = x - l; \;\; \& \;\; x = a\sqrt[m-1]{\frac{a}{l}}.$$

Scholium. Quemadmodum ſolutio ſemper poſſibilis eſt pro ellipſi, ſi fuerit $l > a$; ita etiam ſemper poſſibilis eſt pro hyperbola, ſi fuerit $l \begin{smallmatrix} < a \\ > 0 \end{smallmatrix}$.

§. 45. Cum expreſſio ſubtangentis t ſit $y\frac{dx}{dy}$: ſi ſubtangens data fuerit per quantitatem conſtantem, vel per functionem quantitatum x & y; ideo datus erit exponens differentialis $\frac{dx}{dy}$ per functiones earumdem variabilium, & proinde determinatio quantitatum x & y pertinebit ad calculum integralem.

Exempla. Sit $t = \frac{m}{n}y$; hinc $y\frac{dx}{dy} = \frac{m}{n}y$; $\frac{dx}{dy} = \frac{m}{n}$; $x = C + \frac{m}{n}$, quæ eſt æquatio ad lineam rectam.

Sit $t = \frac{y^m}{a^{m-1}}$; $y\frac{dx}{dy} = \frac{y^m}{a^{m-1}}$; $\frac{dx}{dy} = \frac{y^{m-1}}{a^{m-1}}$; $x = C + \frac{1}{m}\frac{y^m}{a^{m-1}}$. Sit $x = 0$, quando $y = 0$: tum $C = 0$; $x = \frac{1}{m} \cdot \frac{y^m}{a^m}$. Sit vero $x = b$, quando $y = 0$: erit $C = b$, & $x - b = \frac{1}{m} \cdot \frac{y^m}{a^{m-1}}$.

Sit $t = \frac{x^m y^n}{a^{m+n-1}}$; unde $\frac{x^m y^n}{a^{m+n-1}} = y\frac{dx}{dy}$; $\frac{y^{n-1}}{a^{m+n-1}} = \frac{1}{x^m}\frac{dx}{dy} = x^{-m}\frac{dx}{dy}$;

$\frac{\frac{1}{n}y^n}{a^{m+n-1}} = C - \frac{1}{m-1}x^{-m+1}$; $y^n = C - \frac{n}{m-1} \cdot \frac{a^{m+n-1}}{x^{m-1}}$.

Ob

Ob calculi autem integralis imperfectionem plerumque huic problemati (quod *methodus tangentium inverſa* vocatur) æquatione terminis finitis expreſſa ſatisfieri nequit. Imo ſunt caſus nonnulli etiam ſimpliciſſimi, qui methodis huc usque expoſitis tractari non poſſe videntur, & de quibus inferius dicemus. Exempli cauſa ſit $t = \frac{x^m}{a^{m-1}}$: erit $\frac{x^m}{a^{m-1}} = y\frac{dx}{dy}$; de qua formula, & aliis ipſi analogis, infra agemus.

§. 46. Quæcunque ſit origo curvæ alicujus algebraicæ, ea ſemper poteſt ad æquationem reduci inter rectas coordinatas reſpectu alicujus diametri. Ex. gr. conſtantia ſummæ diſtantiarum cujusvis puncti ellipſeos ab ejus focis pro fundamentali ipſius proprietate aſſumta, ex illa deducitur relatio coordinatarum alterutrius axium imo & reſpectu duarum quarumlibet diametrorum conjugatarum: quod ſcilicet quadrata rectarum diametro cuicunque ordinatim applicatarum proportionalia ſint rectangulis ſub abſciſſis ejusdem diametri. Idem valet de hyperbola; pariterque de omnibus ſectionibus conicis, proprietatis fundamentalis loco ſumta ratione data diſtantiarum cujusvis puncti ab alterutro foco & a directrice. Proinde, quæ jam præcepta fuerunt de ductu tangentium, quatenus curvæ ad aliquam diametrum referuntur per rectas huic ordinatim applicatas, ſufficere poſſunt (pro curvis ſaltem algebraicis). Quoniam autem frequenter evenit, ut curvarum affectiones, & nominatim proprietates earum ad contactus pertinentes, modo lucidiori & ſimpliciori ex alia quadam ipſarum origine aut proprietate deduci poſſint, quam ſi æquatio inter coordinatas rectilineas inſtituatur; e re eſſe cenſeo, præcipuas curvarum origines perpendere, & methodos tangentes ducendi iis applicare, non adhibita æquatione, qua curvæ ad diametrum aliquam per rectas huic ordinatim applicatas referuntur. Sed ut hunc ſcopum modo ſufficiente adimplere valeam: nonnulla præmittenda ſunt lemmata, quæ & in ſequentibus capitibus ſuas habebunt applicationes.

§. 47. *Theorema.* Ratio æqualitatis limes eſt tam rationis decreſcentis arcus (qui totus verſus chordam concavus ſupponitur) ad chordam; quam rationis creſcentis arcus ad ſummam tangentium per extrema ejus ductarum, & concurſu ſuo mutuo terminatarum.

Sit

Sit AMZ arcus curvæ (totus verſus chordam ſuam concavus). Dico: ex uno arcus illius extremo A duci poſſe chordam AM talem, ut ratio arcus ALM ad chordam AM (quæ ſemper major eſt ratione æqualitatis) minor ſit quacunque ratione propoſita majoris ad minorem; item, ut ductis ex A & M tangentibus, quæ ſibi mutuo occurrant in N, ratio arcus AM ad ſummam $AN + NM$ duarum tangentium (quæ ſemper minor eſt ratione æqualitatis) major ſit quacunque ratione propoſita minoris ad majorem. Fig. 14.

Sit ratio propoſita majoris ad minorem ea, quam ſumma crurum alicujus trianguli æquicruri $CE + ED$ habet ad baſim CD; ſeu ratio minoris ad majorem ea, quam habet CD ad $CE + ED$. Ad punctum A ducatur tangens BA, & chorda AZ, quæ faciat cum tangente BA angulum BAZ ipſo CED majorem. Tum (§. 43.) ducatur arcui AMZ tangens chordæ AZ parallela, quæ ipſi BA occurret. Sit ea MN. Dico factum.

Etenim ſuper CD, tanquam baſi, conſtituatur triangulum $CE'D$ ipſi ANM ſimile.

In triangulo $CE'D$ angulus $CE'D (= N = BAZ) > CED$. Ergo punctum E' ſitum eſt intra ſegmentum circuli, cujus baſis CD, & quod capax eſt anguli CED. Proinde $CE' + E'D < CE + ED$ (quod facile demonſtratur).

Eſt ideo $CE' + E'D : CD < CE + ED : CD$

ſed $CE' + E'D : CD = AN + NM : AM$

ergo $AN + NM : AM < CE + ED : CD$.

Atqui 1°. arcus ALM minor eſt ſumma $AN + NM$; ergo (a fortiori)

$$ALM : AM < CE + ED : CD$$

2°. arcus ALM major eſt chorda AM; ergo (a fortiori)

$$AN + NM : ALM < CE + ED : CD.$$

Proinde (§. 1.) ratio æqualitatis limes eſt rationis decreſcentis cujusvis arcus ad ſuam chordam; & limes rationis creſcentis ejusdem arcus ad ſummam tangentium, quæ ab extremis arcus ducuntur, & occurſu ſuo mutuo terminantur.

§. 48. *Corollarium* 1. Nominatim ratio æqualitatis limes eſt rationis de-

cre-

creſcentis arcus circuli ad ſuam chordam; & proinde etiam dimidii arcus ad dimidam chordam, ſeu arcus ad ſinum rectum.

Corollarium 2. Rurſus ratio æqualitatis limes eſt rationis creſcentis arcus circuli ad tangentem ejus trigonometricam.

Ceterum corollaria hæc de circulo potuiſſent ex propoſitionibus ARCHIMEDIS de figuris ordinatis, quæ circulo inſcribuntur & circumſcribuntur, deduci.

Corollarium 3. Poſito igitur ſinu alicujus arcus s, ipſe arcus functio eſt ſinus hujus formæ (quæ deinceps accuratius determinabitur)
$s + As^2 + Bs^3 + Cs^4 + Ds^5 + \ldots\ldots$ ideoque ratio arcus circuli ad ſinum rectum eſt $1 + As + Bs^2 + Cs^3 + Ds^4 + \ldots : 1$ & ratio æqualitatis eſt limes hujus rationis decreſcentis (§. 18.)

Item ſit t tangens trigonometrica alicujus arcus circuli, erit arcus functio tangentis hujus formæ $t - At^2 + Bt^3 + Ct^4 + Dt^5 + \ldots\ldots$ Atque ratio arcus circuli ad tangentem trigonometricam $1 - At + Bt^2 + Ct^3 + Dt^4 + \ldots : 1.$ Hujus rationis creſcentis limes eſt ratio æqualitatis (§. 18.)
Viciſſim, poſito arcu circuli z, erit

$$t = z + Az^2 + Bt^3 + Ct^3 + Dt^5 + \ldots\ldots$$

$$t : z = 1 + Az + Bt^2 + Ct^3 + Dt^4 + \ldots\ldots : 1.$$

$$s = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + \ldots\ldots$$

$$s : z = 1 - Az + Bz^2 + Cz^3 + Dz^4 + \ldots\ldots : 1.$$

Corollarium 4. Quoniam ſin. verſ. $z = 2ss$, erit ſin. verſ. $z = A'zz + B'z^3 + C'z^4 + D'z^5 + \ldots\ldots$ Sed hæc ſatius eſt ſequenti modo deducere.

Ratio chordæ ad ſinum verſum poteſt fieri major quacunque ratione data.

Etenim ratio radii dupli ad chordam æqualis eſt rationi chordæ ad ſinum verſum: ſed prior ratio major reddi poteſt quacunque ratione data; ergo & poſterior.

Hinc etiam rationes arcus & ſinus recti ad ſinum verſum poſſunt reddi majores quacunque ratione data. Et ratio æqualitatis limes eſt rationum, quas ſinus rectus, chorda, arcus - - - habent ad ſummam aut differentiam quantitatum harum & ſinus verſi (imo & quantitatum, quæ ad ſinum verſum datas habent rationes §. 18.)

§. 49.

§. 49. *Generatim* per punctum quodlibet M curvæ alicujus ducatur tangens MT, & recta quævis MP; per aliud quodcunque punctum M' ejusdem curvæ ducatur recta alicui rectæ poſitione datæ parallela, quæ ipſis MP, MT occurrat in punctis Q & Q'. Dico: rationem æqualitatis limitem eſſe arcus MNM' & ſegmenti tangentis MQ'. Fig. 11.

Etenim per punctum quodcunque rectæ MP ducatur recta ipſi $M'Q$ parallela, cui chorda MM' & tangens MQ' occurrant in S & T.

$$\text{lim.}\,MNM' : MM' = 1 : 1 \quad (\text{§. 47.})$$
ſed $$\text{lim.}\,MM' : MQ'(=MS:TS) = 1 : 1 \quad (\text{§. 40.})$$
ergo $$\text{lim.}\,MNM' : MQ' = 1 : 1 \quad (\text{§. 14.})$$
item $$\text{lim.}\,QQ' : QM'(=PT:PS) = 1 : 1 \quad (\text{§. 40.})$$

Corollaria. Sit Q angulus rectus.

1°. $$MQ' : MQ = 1 : \text{coſ.}\,TMP$$
$$\text{lim.}\,MNM' : MQ' = 1 : 1$$
ergo $$\text{lim.}\,MNM' : MQ = 1 : \text{coſ.}\,TMP \quad (\text{§. 14.})$$

Scilicet: ratio differentialis arcus cujuslibet curvæ & rectæ axi ſub angulo recto ordinatim applicatæ, æqualis eſt rationi radii ad coſinum anguli, ſub quo tangens curvæ ad rectam axi ordinatim applicatam inclinatur.

2°. $$\text{lim.}\,MNM' : MQ' = 1 : 1$$
$$MQ' : QQ' = 1 : \text{ſin.}\,M$$
$$\text{lim.}\,QQ' : M'Q = 1 : 1$$
ergo $$\text{lim.}\,MNM' : M'Q = 1 : \text{ſin.}\,M \quad (\text{§. 14.})$$

Scilicet: ratio differentialis arcus cujuslibet curvæ & abſciſſæ axis æqualis eſt rationi radii ad ſinum anguli, quem tangens facit cum recta axi ordinatim applicata.

3°. Curva referatur ad aliquod punctum F per radios vectores MF, $M'F$. Fig. 15.
Sit arcus MM'; & radio vectori FM perpendicularis MQ, quæ tangenti in Q' occurrat. Centro F, radio FM', deſcribatur arcus circuli $M'q$. Item ſit PP' arcus, cujus radius FA conſtans.

$$\begin{array}{rlcl}
 & \lim. MNM' : MQ' & = & 1 : 1 \\
 & MQ' : QQ' & = & 1 : \text{ſin.}\, M \\
 & \lim. QQ' : M'Q & = & 1 : 1 \\
 & \lim. MQ : M'q & = & 1 : 1 \qquad (\S.\ 40.) \\
 & M'q : PP' & = & FM : FA \qquad (\S.\ 48.) \\
\text{ergo} & \lim. MNM' : PP' & = & FM : FA\,\text{ſin.}\, M. \qquad (\S.\ 14.)
\end{array}$$

Hoc eſt: ratio differentialis arcus curvæ & arcus circuli, qui metitur angulum duobus radiis vectoribus comprehenſum, æqualis eſt rationi radii vectoris ad radium prædicti circuli, multiplicatum per ſinum anguli, ſub quo tangens curvæ ad radium vectorem inclinatur.

$$\begin{array}{rlcl}
4°. & \lim. MNM : MQ' & = & 1 : 1 \\
 & MQ' : MQ & = & 1 : \text{coſ.}\, M \\
 & \lim. MQ : Mq & = & 1 : 1 \qquad (\S.\ 48.) \\
\text{ergo} & \lim. MNM' : Mq & = & 1 : \text{coſ.}\, M.
\end{array}$$

Hoc eſt: ratio differentialis arcus alicujus curvæ & radii vectoris æqualis eſt rationi radii ad coſinum anguli, quem tangens curvæ cum radio vectore comprehendit.

$$\begin{array}{rlcl}
5°. & \lim. Mq : MQ & = & 1 : 1 \qquad (\S.\ 48.) \\
 & \lim. MQ : M'Q & = & 1 : \text{tang.}\, M \\
 & \lim. M'Q : M'q & = & 1 : 1 \qquad (\S.\ 48.) \\
 & \lim. M'q : PP' & = & FM : FA \\
\text{ergo} & \lim. Mq : PP' & = & FM : FA\,\text{tang.}\, M \qquad (\S.\ 14.)
\end{array}$$

Hoc eſt: ratio differentialis radii vectoris & arcus circuli, qui metitur angulum duobus radiis vectoribus comprehenſum, æqualis eſt rationi radii vectoris ad radium hujus circuli multiplicatum per tangentem anguli, quem tangens curvæ cum radio vectore comprehendit.

Tranſeo ad applicationes.

§. 50. 1°. Curva referatur ad plures rectas poſitione datas per aliquam relationem datam inter perpendiculares, in rectas poſitione datas ex quolibet curvæ puncto demiſſas.

Rectæ

Rectæ perpendiculares in rectas poſitione

datas demiſſæ, dicantur	MP,	MP',	MP'',	MP''',	MP^{iv} ....
ſeu	y,	y',	y'',	y''',	y^{iv} ....
Arcus curvæ dicatur z. Ex relatione data dabitur etiam relatio exponentium differentialium	$\frac{dy}{dz}$,	$\frac{dy'}{dz}$,	$\frac{dy''}{dz}$,	$\frac{dy'''}{dz}$,	$\frac{dy^{\text{iv}}}{dz}$ ....

proinde etiam dabitur relatio inter quantitates $\quad \text{coſ.}\,TMP,\ \text{coſ.}\,TMP',\ \text{coſ.}\,TMP'',\ \text{coſ.}\,TMP''',\ \text{coſ.}\,TMP^{\text{iv}} \ldots$

Atqui punctum M & perpendicula MP, MP', MP'', MP''', MP^{iv}.... dantur poſitione; ergo dabitur aliqua affectio tangentis, per quam determinabitur.

Exemplum 1. Summa rectangulorum prædictorum perpendiculorum per rectas magnitudine datas a, a', a'', a''', a^{iv}.... datur magnitudine.

Erit ideo

$$a\,\text{coſ.}\,TMP + a'\,\text{coſ.}\,TMP' + a''\,\text{coſ.}\,TMP'' + a'''\,\text{coſ.}\,TMP''' + a^{\text{iv}}\,\text{coſ.}\,TMP^{\text{iv}} + \ldots = 0.$$

Locus propoſitus non eſt curva aliqua, ſed linea recta (uti videre eſt ex opuſculo meo, *Polygonometrie*, Geneve 1789.)

Exemplum 2. Summa ſpatiorum, quæ ad quadrata perpendiculorum datas habent rationes, datur magnitudine.

Erit ideo

$$ay\,\text{coſ.}\,TMP + a'y'\,\text{coſ.}\,TMP' + a''y''\,\text{coſ.}\,TMP'' + a'''y'''\,\text{coſ.}\,TMP''' + a^{\text{iv}}y^{\text{iv}}\,\text{coſ.}\,TMP^{\text{iv}} + \ldots = 0.$$

Quam proprietatem ad ſectiones conicas pertinere demonſtrare poſſum.

Exemplum 3. Summa rectangulorum prædictis perpendiculis binis ſumptis factorum datur magnitudine.

Fit ideo

$$\left.\begin{array}{llllll}
 & y'\,\text{coſ.}\,TMP & + y''\,\text{coſ.}\,TMP & + y'''\,\text{coſ.}\,TMP & + y^{\text{iv}}\,\text{coſ.}\,TMP & + \ldots \\
y\,\text{coſ.}\,TMP' & - & + y''\,\text{coſ.}\,TMP' & + y'''\,\text{coſ.}\,TMP' & + y^{\text{iv}}\,\text{coſ.}\,TMP' & + \ldots \\
y\,\text{coſ.}\,TMP'' & + y'\,\text{coſ.}\,TMP'' + & - & + y'''\,\text{coſ.}\,TMP'' & + y^{\text{iv}}\,\text{coſ.}\,TMP'' & + \ldots \\
y\,\text{coſ.}\,TMP''' & + y'\,\text{coſ.}\,TMP''' & + y''\,\text{coſ.}\,TMP''' + & - & + y^{\text{iv}}\,\text{coſ.}\,TMP''' & + \ldots \\
y\,\text{coſ.}\,TMP^{\text{iv}} & + y'\,\text{coſ.}\,TMP^{\text{iv}} & + y''\,\text{coſ.}\,TMP^{\text{iv}} & + y'''\,\text{coſ.}\,TMP^{\text{iv}} + & - & + \ldots \\
- & - & - & - & - & - \\
- & - & - & - & - & -
\end{array}\right\} = 0$$

quæ proprietas ad ſectiones conicas pertinet.

 §. 51.

§. 51. 2°. Sit curva ad plura puncta relata, per datam relationem inter radios vectores ad hæc puncta ductos.

Puncta data ſint	F,	F',	F'',	F''',	F^{iv} . . . .
Radii vectores dicantur	r,	r',	r'',	r''',	r^{iv} . . . .
Ex relatione data ducatur relatio inter exponentes differentiales	$\frac{dr}{dz}$,	$\frac{dr'}{dz}$,	$\frac{dr''}{dz}$,	$\frac{dr'''}{dz}$,	$\frac{dr^{iv}}{dz}$ . . . .

ſeu inter quantitates cof.FMT, cof.$F'MT$, cof.$F''MT$, cof.$F'''MT$, cof.$F^{iv}MT$....
unde poſitio tangentis determinatur.

Exemplum 1. Sint duo puncta data; & ſumma radiorum rectorum detur magnitudine.

Quoniam $r + r'$ datur magnitudine;

eſt $\frac{dr}{dz} + \frac{dr'}{dz} = 0$

ſeu cof.FMT + cof.$F'MT = 0$

hinc cof.FMT = cof.180° — $F'MT$, quæ eſt proprietas nota ellipſeos.

Exemplum 2. Differentia radiorum vectorum detur magnitudine.

Quoniam $r - r'$ datur magnitudine;

fit $\frac{dr}{dz} - \frac{dr'}{dz} = 0$

cof.FMT—cof.$F'MT = 0$

$FMT = F'MT$, quæ eſt proprietas nota hyperbolæ.

Exemplum 3. Ratio radiorum vectorum ſit ſemper eadem.

Quoniam ratio $r : r'$ eſt ſemper eadem,

eſt etiam ratio $\frac{dr}{dz} : \frac{dr'}{dz}$ ſemper eadem;

hinc ratio cof.FMT : cof.$F'MT$ ſemper eadem, quæ eſt proprietas circuli.

Exemplum 4. Rectangulum ex radiis vectoribus detur magnitudine.

Quoniam rr' datur magnitudine; $r'\frac{dr}{dz} + r\frac{dr'}{dz} = 0$

r'cof.FMT + r cof.$FMT = 0$

r'cof.$FMT = r$ cof. 180° — $F'MT$, quæ eſt proprietas curvæ Caſſinianæ.

Exem-

Exemplum 5. Summa ſpatiorum, quæ datas habent rationes ad quadrata ex quotlibet radiis vectoribus, detur magnitudine.

Quoniam

$$ar^2 + a'r'^2 + a''r''^2 + a'''r'''^2 + a^{\text{IV}}r^{\text{IV}2} + \ldots$$ datur magnitudine; fit

$$2ar\frac{dr}{dz} + 2a'r'\frac{dr'}{dz} + 2a''r''\frac{dr''}{dz} + 2a'''r'''\frac{dr'''}{dz} + 2a^{\text{IV}}r^{\text{IV}}\frac{dr^{\text{IV}}}{dz} + \ldots = 0.$$

hinc

$$ar\,\text{coſ.}FMT + a'r'\,\text{coſ.}F'MT + a''r''\,\text{coſ.}F''MT + a'''r'''\,\text{coſ.}F'''MT + a^{\text{IV}}r^{\text{IV}}\,\text{coſ.}F^{\text{IV}}MT + \ldots = 0.$$

quæ eſt proprietas circuli. (Vid. *De relatione mutua capacitatis et terminorum figurarum.* Vars. 1782.)

§. 52. 3°. Curva referatur ad aliquod punctum datum, & ad rectam, per angulum, quem radius vector facit cum recta poſitione data.

Angulus mutabilis dicatur x.

Ex relatione data inter radium vectorem & angulum mutabilem dabitur exponens differentialis $\frac{dr}{dx}$, ſeu $r\,\text{cot.}\,FMT$.

Exemplum 1. Sit radius vector r datus magnitudine.

Fit ideo $\frac{dr}{dx} = 0$; $\text{coſ.}FMT = 0$, $FMT = 90°$, quæ eſt proprietas circuli.

Exemplum 2. Sit $r = ax$

$\frac{dr}{dx} = a$; $r\,\text{cot.}FMT = a$; $\text{tang.}FMT = \frac{r}{a}$, quæ eſt proprietas ſpiralis Archimedeæ.

Exemplum 3. Sit $r\,\text{coſ.}^2\tfrac{1}{2}x = a$.

$$\text{Erit}\quad \frac{dr}{dx}\text{coſ.}^2\tfrac{1}{2}x - r\,\text{coſ.}\tfrac{1}{2}x\,\text{ſin.}\tfrac{1}{2}x = 0$$

$$r\,\text{cot.}FMT\,\text{coſ.}\tfrac{1}{2}x = r\,\text{ſin.}\tfrac{1}{2}x$$

$$\text{cot.}FMT = \text{tang.}\tfrac{1}{2}x$$

$$FMT = 90° - \tfrac{1}{2}x,$$ quæ eſt proprietas parabolæ.

Exemplum 4. Sit $r = \frac{bb}{a+e\,\text{coſ.}x}$

$$\text{erit}\quad \frac{dr}{dx} = -\frac{bbe\,\text{ſin.}x}{(a+e\,\text{coſ.}x)^2}$$

$$r\,\text{cot.}FMT = \frac{bbe\,\text{ſin.}x}{(a+e\,\text{coſ.}x)^2} = r\,.\,\frac{e\,\text{ſin.}x}{a+e\,\text{coſ.}x}$$

$$\text{cot.}FMT = \frac{e\,\text{ſin.}x}{a+e\,\text{coſ.}x},$$ quæ eſt proprietas ellipſeos.

§. 53. 4°. Curva referatur ad aliquod punctum & ad aliquam rectam, aut etiam ad plura puncta ſimul & ad plures rectas.

Ex relatione data inter quantitates

$$r, \quad r', \quad r'', \quad r''', \ldots \quad y, \quad y', \quad y'', \quad y''' \ldots$$

dabitur relatio inter exponentes differentiales

$$\frac{dr}{dz}, \quad \frac{dr'}{dz}, \quad \frac{dr''}{dz}, \quad \frac{dr'''}{dz}, \ldots \quad \frac{dy}{dz}, \quad \frac{dy'}{dz}, \quad \frac{dy''}{dz}, \quad \frac{dy'''}{dz} \ldots$$

ſeu inter quantitates

$$\text{coſ.}FMT, \text{coſ.}F'MT, \text{coſ.}F''MT, \text{coſ.}F'''MT \ldots \text{coſ.}PMT, \text{coſ.}P'MT, \text{coſ.}P''MT, \text{coſ.}P'''MT \ldots$$

unde poſitio tangentis determinabitur.

1°. Sit unicum punctum & unica recta.

Exemplum 1. Sit $r^2 = ay$

$$\text{fit} \quad 2r\frac{dr}{dz} = a\frac{dy}{dz}$$

$2r\,\text{coſ.}FMT = a\,\text{coſ.}PMT$, quæ proprietas pertinet ad circulum.

Exemplum 2. Sit $r = \frac{m}{n}y$

$$\text{fit} \quad \frac{dr}{dz} = \frac{m}{n}\frac{dy}{dz}$$

$\text{coſ.}FMT = \frac{m}{n}\,\text{coſ.}PMT$, quæ proprietas pertinet ad ſectiones conicas.

2°. Sint plura puncta & plures rectæ.

Exemplum.

$$\text{Sit} \quad mr^2 + m'r'^2 + m''r''^2 + m'''r'''^2 + \ldots$$
$$= ay + a'y' + a''y'' + a'''y''' + \ldots$$

$$\text{erit} \quad 2mr\frac{dr}{dz} + 2m'r'\frac{dr'}{dz} + 2m''r''\frac{dr''}{dz} + 2m'''r'''\frac{dr'''}{dz} + \ldots$$
$$= a\frac{dy}{dz} + a'\frac{dy'}{dz} + a''\frac{dy''}{dz} + a'''\frac{dy'''}{dz} + \ldots$$

$$\text{ſeu} \quad 2mr\,\text{coſ.}FMT + 2m'r'\,\text{coſ.}F'MT + 2m''r''\,\text{coſ.}F''MT + 2m'''r'''\,\text{coſ.}F'''MT + \ldots$$
$$= a\,\text{coſ.}PMT + a'\,\text{coſ.}P'MT + a''\,\text{coſ.}P''MT + a'''\,\text{coſ.}P'''MT + \ldots$$

quæ proprietas pertinet ad circulum.

Scholium.

Scholium. Ex his exemplis patet, ſæpe potius eſſe, proprietates curvarum ex proprietate quapiam primaria deducere; operationesque mere algebraicas poſſe fieri & longiores & minus lucidas, quam quæ magis geometrice inſtituuntur.

Caput Sextum.

De Logarithmis.

§. 54.

In diſſertatione jam nominata *Expoſition elementaire des principes des calculs ſuperieurs* cap. VI. logarithmorum theoriam & calculum ex contemplatione curvæ logarithmicæ deduxi. Et quidem curva hæc omnino mihi apta videtur ad dilucidandum hoc caput, quod tanti eſt per univerſam matheſin momenti. Cum vero idem ſpectari poſſit tanquam ad calculum proprie pertinens, methodus mere algebraica, & progreſſionum geometricarum natura immediate nixa, potior videri poteſt, quam methodus mixta (nempe partim geometrica & partim algebraica), qua in prædicta diſſertatione uſus ſum.

Methodus autem, quam hic proſecuturus ſum, eſt mere elementaris, neque ulla infiniti notione affecta. Eandemque eo lubentius evolvam, cum (quod ſciam) prorſus ſit nova, principiis prius poſitis omnino conſentanea, fæcunditati eorum luculenter oſtendendæ apta, & demonſtrationi Pfleidererianæ theorematis Tayloriani valde analoga.

§. 55. Sit a^z quantitas exponentialis, quæ proinde functio eſt exponentis mutabilis z. Et quoniam imminuta z (poſito a numero poſitivo unitate majore) unitas eſt limes quantitatis decreſcentis a^z (§. 22.)

Sit $a^z = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + Gz^7 + \ldots$

Erit ideo

$$a^{2z} = 1+2Az+2^2Bz^2+2^3Cz^3+2^4Dz^4+2^5Ez^5+2^6Fz^6+2^7Gz^7+\ldots$$

$$a^{3z} = 1+3Az+3^2Bz^2+3^3Cz^3+3^4Dz^4+4^5Ez^5+3^6Fz^6+3^7Gz^7+\ldots$$

$$a^{4z} = 1+4Az+4^2Bz^2+4^3Cz^3+4^4Dz^4+4^5Ez^5+4^6Fz^6+4^7Gz^7+\ldots$$

$$a^{5z} = 1+5Az+5^2Bz^2+5^3Cz^3+5^4Dz^4+5^5Ez^5+5^6Fz^6+5^7Gz^7+\ldots$$

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Hinc

Hinc ſumtis differentiis primis erit (§. 5. *Introd.*)

$$(a^z-1)a^z = Az+(2^2-1)Bz^2+(2^3-1)Cz^3+(2^4-1)Dz^4+(2^5-1)Ez^5+(2^6-1)Fz^6+\dots$$
$$(a^z-1)a^{2z}=Az+(3^2-2^2)Bz^2+(3^3-2^3)Cz^3+(3^4-2^4)Dz^4+(3^5-2^5)Ez^5+(3^6-2^6)Fz^6+\dots$$
$$(a^z-1)a^{3z}=Az+(4^2-3^2)Bz^2+(4^3-3^3)Cz^3+(4^4-3^4)Dz^4+(4^5-3^5)Ez^5+(4^6-3^6)Fz^6+\dots$$
$$(a^z-1)a^{4z}=Az+(5^2-4^2)Bz^2+(5^3-4^3)Cz^3+(5^4-4^4)Dz^4+(5^5-4^5)Ez^5+(5^6-4^6)Fz^6+\dots$$

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Quoniam autem $a^z = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \dots$

et $a^z - 1 = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \dots$

primus terminus ſingulorum productorum, $(a^z-1)a^z$, $(a^z-1)a^{2z}$, $(a^z-1)a^{3z}$, $(a^z-1)^{4z}$, ... ipſe fit Az: unde in æquationibus præcedentibus dividendo per z, & æquando terminos conſtantes (juxta methodum reverſionis ſerierum), fit $A=A$. Quare A manet quantitas indeterminata.

Sumantur differentiæ ſecundæ. Erit (§. 5. *Introd.*)

$$(a^z-1)^2a^z = \Delta''(3^2\dots1^2)Bz^2+\Delta''(3^3\dots1^3)Cz^3+\Delta''(3^4\dots1^4)Dz^4+\Delta''(3^5\dots1^5)Ez^5+\dots$$
$$(a^z-1)^2a^{2z} = \Delta''(4^2\dots2^2)Bz^2+\Delta''(4^3\dots2^3)Cz^3+\Delta''(4^4\dots2^4)Dz^4+\Delta''(4^5\dots2^5)Ez^5+\dots$$
$$(a^z-1)^2a^{3z} = \Delta''(5^2\dots3^2)Bz^2+\Delta''(5^3\dots3^3)Cz^3+\Delta''(5^4\dots3^4)Dz^4+\Delta''(5^5\dots3^5)Ez^5+\dots$$
$$(a^z-1)^2a^{4z} = \Delta''(6^2\dots4^2)Bz^2+\Delta''(6^3\dots4^3)Cz^3+\Delta''(6^4\dots4^4)Dz^4+\Delta''(6^5\dots4^5)Ez^5+\dots$$

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Jam vero, quoniam $a^z-1 = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \dots$ primus terminus tam quadrati $(a^z-1)^2$, quam ſingulorum productorum hujus quadrati per terminos a^z, a^{2z}, a^{3z}, a^{4z} . . . . eſt AAz^2. Coëfficiens autem primi termini Bz^2 ſecundi membri cujusvis æquationum præcedentium eſt differentia ſecunda quadratorum numerorum naturalium; & proinde (§. 9. *Introd.*) quantitas conſtans 1.2. Hinc dividendo utrinque per z^2, & æquando invicem terminos conſtantes (juxta methodum reverſionis ſerierum), fit $AA = 1.2B$; & proinde $B = \frac{1}{1.2}AA$.

Suman-

Sumantur differentiæ tertiæ; erit (§. *s. Introd.*)

$$(a^z-1)^3a^z \;=\Delta'''(4^3\ldots1^3)Cz^3+\Delta'''(4^4\ldots1^4)Dz^4+\Delta'''(4^5\ldots1^5)Ez^5+\Delta'''(4^6\ldots1^6)Fz^6+\ldots$$
$$(a^z-1)^3a^{2z}=\Delta'''(5^3\ldots2^3)Cz^3+\Delta'''(5^4\ldots2^4)Dz^4+\Delta'''(5^5\ldots2^5)Ez^5+\Delta'''(5^6\ldots2^6)Fz^2+\ldots$$
$$(a^z-1)^3a^{3z}=\Delta'''(6^3\ldots3^3)Cz^3+\Delta'''(6^4\ldots3^4)Dz^4+\Delta'''(6^5\ldots3^5)Ez^5+\Delta'''(6^6\ldots3^6)Fz^6+\ldots$$
$$(a^z-1)^3a^{4z}=\Delta'''(7^3\ldots4^3)Cz^3+\Delta'''(7^4\ldots4^4)Dz^4+\Delta'''(7^5\ldots4^5)Ez^5+\Delta'''(7^6\ldots4^6)Fz^6+\ldots$$

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Quoniam $a^z - 1 = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \ldots\ldots$
primus terminus tam cubi $(a^z-1)^3$, quam ſingulorum productorum prædicti cubi per terminos a^z, a^{2z}, a^{3z}, a^{4z}. . . . eſt A^3z^3.

Coëfficiens autem primi termini Cz^3 ſecundi membri cujusvis æquationum præcedentium eſt differentia tertia cuborum numerorum naturalium; & proinde (§. *q. Introd.*) quantitas conſtans $=1.2.3$: hinc eadem methodo, qua coëfficiens B fuit determinatus, fit $A^3 = 1.2.3C$; & $C = \frac{1}{1.2.3}A^3$.

Sumantur differentiæ quartæ; erit (§. *s. Introd.*)

$$(a^z-1)^4a^z \;=\Delta^{\text{iv}}(5^4\ldots1^4)Dz^4+\Delta^{\text{iv}}(5^5\ldots1^5)Ez^5+\Delta^{\text{iv}}(5^6\ldots1^6)Fz^6+\Delta^{\text{iv}}(5^7\ldots1^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{2z}=\Delta^{\text{iv}}(6^4\ldots2^4)Dz^4+\Delta^{\text{iv}}(6^5\ldots2^5)Ez^5+\Delta^{\text{iv}}(6^6\ldots2^6)Fz^6+\Delta^{\text{iv}}(6^7\ldots2^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{3z}=\Delta^{\text{iv}}(7^4\ldots3^4)Dz^4+\Delta^{\text{iv}}(7^5\ldots3^5(Ez^5+\Delta^{\text{iv}}(7^6\ldots3^6)Fz^6+\Delta^{\text{iv}}(7^7\ldots3^7)Gz^7+\ldots$$
$$(a^z-1)^4a^{4z}=\Delta^{\text{iv}}(8^4\ldots4^4)Dz^4+\Delta^{\text{iv}}(8^5\ldots4^5)Ez^5+\Delta^{\text{iv}}(8^6\ldots4^6)Fz^6+\Delta^{\text{iv}}(8^7\ldots4^7)Gz^7+\ldots$$

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Quoniam $a^z - 1 = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \ldots\ldots$
primus terminus tam quartæ poteſtatis $(a^z-1)^4$, quam ſingulorum productorum hujus poteſtatis per terminos a^z, a^{2z}, a^{3z}, a^{4z}, . . . , eſt A^4z^4.

Coëfficiens autem primi termini Dz^4 ſecundi membri cujusvis æquationum præcedentium eſt differentia quarta quartarum poteſtatum numerorum naturalium; & proinde (§. *q. Introd.*) quantitas conſtans $= 1.2.3.4$. Hinc priore methodo fit $A^4 = 1.2.3.4D$; & $D = \frac{1}{1.2.3.4}A^4$.

Eodem prorſus modo, ſumtis differentiis quintis, primus terminus omnium productorum $(a^z-1)^5a^{nz}$ eſt A^5z^5; & coëfficiens primi termini Ez^5 poſterio-

ris

ris membri ſingularum expreſſionum horum productorum eſt differentia quinta quintarum poteſtatum numerorum naturalium, nempe 1. 2. 3. 4. 5. Hinc $A^5 = 1.2.3.4.5E$; & $E = \frac{1}{1.2\ldots 5} A^5$.

Tum ſumtis differentiis ſextis, primus terminus omnium productorum $(a^z - 1)^6 a^{nz}$, eſt $A^6 z^6$. Coëfficiens autem termini Fz^6 eſt differentia ſexta ſextarum poteſtatum numerorum naturalium; & proinde (§. *q. Introd.*) quantitas conſtans $1.2\ldots 6$. Hinc $A^6 = 1.2\ldots 5.6F$; & $F = \frac{1}{1.2\ldots 5.6} A^6$.

Eadem methodo determinantur coëfficientes potentiarum ſucceſſivarum ipſius z; unde conſequitur formula generalis.

$$a^z (= 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \ldots\ldots)$$
$$= 1 + Az + \frac{1}{1.2} A^2 z^2 + \frac{1}{1.2.3} A^3 z^3 + \frac{1}{1.2..4} A^4 z^4 + \frac{1}{1.2..5} A^5 z^5 + \ldots\ldots$$

§. 56. Quantitas ideo exponentialis a^z exprimi poteſt juxta legem admodum regularem per exponentem z, & aliam quamdam quantitatem A, quam ab ipſa a pendere deinceps videbimus.

Si vero z ſit numerus negativus, erit eodem omnino modo $\frac{1}{a^z} (= a^{-z}) =$

$$= 1 - Az + \frac{1}{1.2} A^2 z^2 - \frac{1}{1.2.3} A^3 z^3 + \frac{1}{1.2\ldots 4} A^4 z^4 - \frac{1}{1.2\ldots 5} A^5 z^5 + \ldots\ldots$$

Obſervatio. Si Az unitate major non eſt: ſeries hæc non modo convergit; ſed etiam termini ipſius, inde a termino quocunque propoſito ſumti, celerius decreſcunt, quam termini progreſſionis alicujus geometricæ decreſcentis.

Etenim ſit $Az = 1$; ſintque tres coëfficientes ſucceſſivi $\frac{1}{1.2\ldots p}$, $\frac{1}{1.2\ldots p+1}$, $\frac{1}{1.2\ldots p+2}$: termini hi ſunt inter ſe, uti 1, $\frac{1}{p+1}$, $\frac{1}{p+1} \cdot \frac{1}{p+2}$. Jam vero $\frac{1}{p+1} \cdot \frac{1}{p+2} < \frac{1}{(p+1)^2}$; ergo termini hi velocius decreſcunt quam in progreſſione geometrica, cujus exponens $\frac{1}{p+1}$. Idemque tanto magis obtinet, ſi $Az < 1$.

Quodſi autem $Az > 1$. Sit p numerus integer non major ipſo Az, ſed illi proximus. Termini Az, $\frac{1}{1.2} A^2 z^2$, $\frac{1}{1.2.3} A^3 z^3$, $\frac{1}{1.2\ldots 4} A^4 z^4 \ldots\ldots \frac{1}{1.2\ldots p} A^p z^p$ continue

tinue creſcunt: qui vero hos ſequuntur termini, continue decreſcunt, & rapidius quidem quam in progreſſione geometrica. Etenim tres termini ſucceſſivi,
$\frac{1}{1.2 \ldots p+m} A^{p+m} z^{p+m}$, $\frac{1}{1.2 \ldots p+m+1} A^{p+m+1} z^{p+m+1}$, $\frac{1}{1.2 \ldots p+m+2} A^{p+m+2} z^{p+m+2}$
ſunt inter ſe uti quantitates 1, $\frac{Az}{p+m+1}$, $\frac{A^2 z^2}{p+m+1 . p+m+2}$. Quoniam autem $\frac{Az}{p+1} < 1$, tanto magis eſt $\frac{Az}{p+m+1} < 1$; pariterque $\frac{A^2 z^2}{p+m+1 . p+m+2} < \frac{A^2 z^2}{(p+m+1)^2}$. Proinde termini ſucceſſivi rapidius decreſcunt quam in progreſſione geometrica, cujus exponens eſt numerus fractus $\frac{Az}{p+m+1}$. Nominatim prædicta ſeries inde a termino $\frac{A^p z^p}{1.2 \ldots p}$ rapidius decreſcit quam in progreſſione geometrica, cujus exponens $\frac{Az}{p+1}$: itaque ſumma terminorum ab hoc termino decreſcentium minor eſt quam $\frac{A^p z^p}{1.2 \ldots p} \left[\frac{1}{1-\frac{Az}{p+1}}\right]$, ſeu $\frac{A^p z^p}{1.2 \ldots p} \cdot \frac{p+1}{p+1-Az}$. Proinde utut magnus ſit numerus Az, ſeries hæc ultra certum limitem non excurrit.

Ceterum quod ad applicationes ſeriei hujus attinet; fatendum, eam uſus facilis eſſe iis tantum caſibus, quibus $Az \overline{\overline{<}}\, 1$.

§. 57. Applicationum quarundam hujus ſeriei gratia, quæ deinceps exponentur, e re erit in ejus naturam penitius inquirere.

Sit binomium $\left(1+\frac{Az}{n}\right)^n$; erit ſemper

$$\left(1+\frac{Az}{n}\right)^n = 1 + \frac{n}{1}\,\frac{Az}{n} + \frac{n}{1}\cdot\frac{n-1}{2}\,\frac{A^2 z^2}{n^2} + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\,\frac{A^3 z^3}{n^3} + \frac{n}{1}\cdots\frac{n-3}{4}\,\frac{A^4 z^4}{n^4} + \ldots$$

$$= 1 + Az + \frac{1-\frac{1}{n}}{1.2} A^2 z^2 + \frac{1-\frac{1}{n}}{1.2}\cdot\frac{1-\frac{2}{n}}{3} A^3 z^3 + \frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{3}{n}}{4} A^4 z^4 + \ldots$$

Sit autem n numerus poſitivus, qui major fieri poſſit quocunque numero propoſito.

1°. Sit $Az = 1$. Unitas eſt limes factorum $1-\frac{1}{n}$, $1-\frac{2}{n}$, $1-\frac{3}{n}$, ... $1-\frac{p}{n}$, quorum numerus p; & proinde etiam unitas eſt limes producti

1 —

$1-\frac{1}{n}.\ 1-\frac{2}{n}.\ 1-\frac{3}{n}.\ 1-\frac{4}{n}.\ \dots\ 1-\frac{p}{n}$ (§. 21.); unde ſeries

$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2\dots 4}+\dots+\frac{1}{1.2\dots p+1}$ eſt limes ſeriei

$1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}+\frac{1-\frac{1}{n}}{1.2}\dots\frac{1-\frac{3}{n}}{4}+\dots\dots\frac{1-\frac{1}{n}}{1.2}\dots\frac{1-\frac{p}{n}}{p+1}.$

Atqui, quo major p, eo minores ſunt factores ſequentes $1-\frac{p+m}{n}$; & eo majores ſunt denominatores $1.2\dots p+m+1$ terminorum ſucceſſivorum; proinde, a fortiori, ſeries $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2..4}+\frac{1}{1.2..5}+\dots$ eſt limes ſeriei $1+1+\frac{1-\frac{1}{n}}{1.2}+\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}+\frac{1-\frac{1}{n}}{1.2}\dots\frac{1-\frac{3}{n}}{4}+\frac{1-\frac{1}{n}}{1.2}\dots\frac{1-\frac{4}{n}}{5}+\dots$; ſeu quantitatis $\left(1+\frac{1}{n}\right)^n$.

2°. Hæc valent a fortiori, ſi $Az < 1$.

3°. Sit $Az > 1$; & ſit p numerus integer non major quam z, ſed ei proximus.

Sumto termino $\frac{1-\frac{1}{n}}{1.2}\dots\frac{1-\frac{p-1}{n}}{p}A^p z^p$; termini, qui illum ſequuntur, rapidius decreſcunt quam in progreſſione geometrica, cujus exponens $\frac{Az}{p+1}$: tum quia factores ſucceſſivi denominatoris $p+2,\ p+3,\ p+4\dots p+m$ majores ſunt factore $p+1$; tum etiam quia factores numeratoris $1-\frac{p}{n},\ 1-\frac{p+1}{n}\dots 1-\frac{p+n-1}{n}$ fiunt continue minores: & proinde, a fortiori, ſumma omnium terminorum, inde a termino $\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}\dots\frac{1-\frac{p-1}{n}}{p}A^p z^p$, minor eſt quantitate $\frac{1-\frac{1}{n}}{1.2}.\frac{1-\frac{2}{n}}{3}\dots\frac{1-\frac{p-1}{n}}{p}A^p z^p\left(\frac{1}{1-\frac{Az}{p+1}}\right)$, cujus limes eſt

$\frac{1}{1.2\dots p}A^p z^p.\frac{1}{1-\frac{Az}{p+1}}$; tantoque magis ſumma omnium terminorum inde a termino

mino $\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m}A^{p+m}z^{p+m}$ minor eſt quantitate

$\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m}A^{p+m}z^{p+m}\left(\frac{1}{1-\frac{Az}{p+m}}\right)$, ſeu $\frac{1-\frac{1}{n}}{1.2}\cdots\frac{1-\frac{p+m-1}{n}}{p+m-1}A^{p+m}z^{p+m}.\frac{1}{p+m-Az}$,

cujus limes eſt $\frac{1}{1.2\ldots p+m-1}A^{p+m}z^{p+m}.\frac{1}{p+m-Az}$; quæ cum minor ſit quantitate $\frac{A^p z^p}{1.2\ldots p}.\frac{Az}{p+m-Az}.\frac{A^{m-1}z^{m-1}}{(p+1)^{m-1}}$, poteſt reddi minor quacunque quantitate propoſita.

Proinde tandem in omni caſu ſeries

$1+Az+\frac{1}{1.2}A^2z^2+\frac{1}{1.2.3}A^3z^3+\frac{1}{1.2\ldots4}A^4z^4+\ldots$ limes eſt ſeriei, quæ oritur ex quantitate $\left(1+\frac{Az}{n}\right)^n$ evoluta, ſeu eſt limes hujus quantitatis.

§. 58. Quoniam a exponens eſt rationis $a:1$; a^z exponens eſt rationis $(a:1)^z$. Scilicet ratio $(a:1)^z$ componitur ex z rationibus ipſi rationi $a:1$ æqualibus; ſeu z metitur rationem $(a:1)^z$. Hinc z dicitur *logarithmus* rationis $(a:1)^z$; ſeu omiſſo conſequente 1, communi & conſtanti, z dicitur logarithmus ipſius a^z, quatenus logarithmus ipſius a eſt unitas. Quantitas a dicitur *baſis* logarithmica; quantitas A autem dicitur *modulus*. Diverſitas ſyſtematum logarithmicorum pendet a baſi a; proindeque etiam a modulo A.

Baſis itaque a modulo ita pendet, ut ſit

$$a=1+A+\frac{1}{1.2}A^2+\frac{1}{1.2.3}A^3+\frac{1}{1.2\ldots4}A^4+\frac{1}{1.2\ldots5}A^5+\frac{1}{1.2\ldots6}A^6+\ldots$$

Ex. gr. Sit $A=1$; erit $a=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2\ldots4}+\frac{1}{1.2\ldots5}+\frac{1}{1.2\ldots6}+\ldots$

$$=2,\ 718\ 281\ 828\ 4\tfrac{5}{6}\tfrac{+}{-}.$$

Syſtema logarithmicum huic ſuppoſitioni reſpondens dicitur *ſyſtema naturale*, ſeu etiam hyperbolicum; atque inter mathematicos convenit, baſim a hujus ſyſtematis per litteram e deſignare.

§. 59. Quoniam

$$a^z = 1 + Az + \frac{A^2}{1.2}z^2 + \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 + \frac{A^5}{1.2..5}z^5 + \frac{A^6}{1.2...6}z^6 + \ldots$$

et $$a^{-z} = 1 - Az + \frac{A^2}{1.2}z^2 - \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 - \frac{A^5}{1.2...5}z^5 + \frac{A^6}{1.2...6}z^6 + \ldots$$

erit $$\frac{a^z + a^{-z}}{2} = 1 + \frac{A^2}{1.2}z^2 + \frac{A^4}{1.2..4}z^4 + \frac{A^6}{1.2...6}z^6 + \frac{A^8}{1.2...8}z^8 + \frac{A^{10}}{1.2..10}z^{10} + \ldots$$

$$\frac{a^z - a^{-z}}{2} = Az + \frac{A^3}{1.2.3}z^3 + \frac{A^5}{1.2..5}z^5 + \frac{A^7}{1.2...7}z^7 + \frac{A^9}{1.2...9}z^9 + \frac{A^{11}}{1.2..11}z^{11} + \ldots$$

Nominatim

$$\frac{e^z + e^{-z}}{2} = 1 + \frac{1}{1.2}z^2 + \frac{1}{1.2..4}z^4 + \frac{1}{1.2...6}z^6 + \frac{1}{1.2...8}z^8 + \frac{1}{1.2..10}z^{10} + \ldots$$

$$\frac{e^z - e^{-z}}{2} = z + \frac{1}{1.2.3}z^3 + \frac{1}{1.2..5}z^5 + \frac{1}{1.2...7}z^7 + \frac{1}{1.2...9}z^9 + \frac{1}{1.2..11}z^{11} + \ldots$$

§. 60. Quoniam

$$a^z = 1 + Az + \frac{1}{1.2}A^2z^2 + \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2...4}A^4z^4 + \frac{1}{1.2...5}A^5z^5 + \ldots.$$

Exponentibus differentialibus ſumendis erit

$$\frac{\mathrm{d}a^z}{\mathrm{d}z} = A + A^2z + \frac{1}{1.2}A^3z^2 + \frac{1}{1.2.3}A^4z^3 + \frac{1}{1.2...4}A^5z^4 + \frac{1}{1.2...5}A^6z^5 + \ldots.$$

$$= A(1 + Az + \frac{1}{1.2}A^2z^2 + \frac{1}{1.2.3}A^3z^3 + \frac{1}{1.2...4}A^4z^4 + \frac{1}{1.2...5}A^5z^5 + \ldots.$$

$= A \times a^z$. Hinc

$$\frac{\mathrm{d}^2a^z}{\mathrm{d}z^2} = A^2 \times a^z$$

$$\frac{\mathrm{d}^3a^z}{\mathrm{d}z^3} = A^3 \times a^z$$

$$\frac{\mathrm{d}^4a^z}{\mathrm{d}z^4} = A^4 \times a^z$$

$$\frac{\mathrm{d}^5a^z}{\mathrm{d}z^5} = A^5 \times a^z$$

$\vdots \qquad \vdots$

Pro-

Proinde exponentes differentiales omnium ordinum quantitatis exponentialis a^z exponentisque ejus z ſequuntur progreſſionem geometricam, cujus exponens eſt ipſi A æqualis.

§. 61. Quoniam $a = 1 + A + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2...4} + \frac{A^5}{1.2...5} + \frac{A^6}{1.2...6} + \ldots$

$$\text{ſeu} \quad a - 1 = A + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2...4} + \frac{A^5}{1.2...5} + \frac{A^6}{1.2...6} + \ldots$$

Adhibita methodo reverſionis ſerierum invenitur modulus A per baſim a expreſſus. Potior autem mihi videtur uſus ſequentis methodi, qua ſimul logarithmorum ipſorum expreſſiones conſequimur.

Qnoniam $a^z = 1 + Az + \frac{A^2}{1.2}z^2 + \frac{A^3}{1.2.3}z^3 + \frac{A^4}{1.2...4}z^4 + \frac{A^5}{1.2...5}z^5 \ldots$

fiat $z = n\Delta z$.

Erit etiam $a^{\Delta z} = 1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots$

$$a^z = (a^{n\Delta z}) = \left(1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots\right)^n$$

Sit autem $\left(1 + A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots\right)^n = 1 + v$

erit $A\Delta z + \frac{A^2}{1.2}\Delta z^2 + \frac{A^3}{1.2.3}\Delta z^3 + \frac{A^4}{1.2..4}\Delta z^4 + \frac{A^5}{1.2..5}\Delta z^5 + \ldots) = (1+v)^{\frac{1}{n}} - 1$

et $An\Delta z\left(1 + \frac{A}{1.2}\Delta z + \frac{A^2}{1.2.3}\Delta z^2 + \frac{A^3}{1.2..4}\Delta z^3 + \frac{A^4}{1.2..5}\Delta z^4 + \ldots\right) = n\left((1+v)^{\frac{1}{n}} - 1\right)$

$$= v - \frac{1-\frac{1}{n}}{1.2}v^2 + \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}v^3 - \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}\cdot\frac{3-\frac{1}{n}}{4}v^4 + \frac{1-\frac{1}{n}}{1.2}\cdot\frac{2-\frac{1}{n}}{3}\cdot\frac{3-\frac{1}{n}}{4}\cdot\frac{4-\frac{1}{n}}{5}v^5 - \ldots$$

Sed $n\Delta z = z = \log. 1 + v$ (hyp.)

Ergo $A \log. 1 + \underline{v}\left(1 + \frac{A}{1.2}\Delta z + \frac{A^2}{1.2.3}\Delta z^2 + \frac{A^3}{1.2..4}\Delta z^3 + \frac{A^4}{1.2..5}\Delta z^4 + \frac{A^5}{1.2..6}\Delta z^5 + \ldots\right)$

$$= v - \frac{1-\frac{\Delta z}{z}}{1.2}v^2 + \frac{1-\frac{\Delta z}{z}}{1.2}\cdot\frac{2-\frac{\Delta z}{z}}{3}v^3 - \frac{1-\frac{\Delta z}{z}}{1.2}\cdot\frac{2-\frac{\Delta z}{z}}{3}\cdot\frac{3-\frac{\Delta z}{z}}{4}v^4 + \frac{1-\frac{\Delta z}{z}}{1.2}\ldots\frac{4-\frac{\Delta z}{z}}{5}v^5 - \ldots)$$

Cum

Cum hæc æquatio ſemper locum habeat, limites etiam membrorum ejus ſunt inter ſe æquales (§. 4.) Sed hi limites ſunt $A \log. 1+v$, &

$v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \ldots$

Ergo $A\log. 1+v = \quad v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \ldots\ldots$

Hinc $A\log. 1-v = -(v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4 + \frac{1}{5}v^5 + \ldots\ldots)$

Unde $A\log. \sqrt{1-vv} = -(v^2 + \frac{1}{2}v^4 + \frac{1}{3}v^6 + \frac{1}{4}v^8 + \ldots\ldots)$

et $A\log. \sqrt{\frac{1+v}{1-v}} = \quad (v. + \frac{1}{3}v^3 + \frac{1}{5}v^5 + \frac{1}{7}v^7 + \ldots\ldots)$

Non pertinet ad ſcopum præſentem, varias evolvere formulas variaque artificia, quibus (quamvis diu poſt inventos & computatos logarithmos) calculi logarithmorum compendia fuere ſubminiſtrata. Conſulantur hac de re ſcriptores, qui artificia illa exponere ſategerunt, quos inter ſagaciſſimus BERTRAND in opere ſuo inſcripto: *Developement nouveau de la partie elementaire des Mathematiques.*

§. 62. Formulam $A\log. 1 + v = v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \ldots\ldots$ conſequi potuiſſemus per prima calculi differentialis principia, ut ſequitur.

Quoniam $\log. a^z = z$; & $\frac{d.a^z}{dz} = A.a^z$; $\frac{d.a^z}{d.\log.z} = A.a^z$; & $\frac{d.\log.z}{d.a^z} = \frac{1}{A} \times \frac{1}{a^z}$.

Sit itaque $y = \log. 1 + v$.

Erit $\frac{dy}{dv} = \frac{1}{A} \cdot \frac{1}{1+v}$; & quoniam $y = 0$; quando $v = 0$, fit (§. 27.)

$y = \frac{1}{A}(v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \ldots)$; quod etiam poteſt obtineri convertendo $\frac{1}{1+v}$ in ſeriem, & integrando.

Nempe: $\frac{dy}{dv} = \frac{1}{A}(1 - v + v^2 - v^3 + v^4 - v^5 + \ldots)$

$y = \frac{1}{A}(v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \frac{1}{5}v^6 + \ldots)$

ſeu $A\log. 1+v = \quad (v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5 - \frac{1}{6}v^6 + \ldots)$.

Corollarium. Quantitate v manente eadem, productum $A\log. 1 + v$ ejusdem etiam magnitudinis manet. Proinde in diverſis ſyſtematibus, logarithmi unius

unius ejusdemque numeri ſunt in ratione inverſa modulorum horum ſyſtematum.

Eſt etiam $A \log. \sqrt{\frac{1+v}{1-v}} = v + \frac{1}{3}v^3 + \frac{1}{5}v^5 + \frac{1}{7}v^7 + \ldots\ldots$

Sit $\frac{1+v}{1-v} = a^{2z}$: unde $v = \frac{a^{2z}-1}{a^{2z}+1}$; & $A \log. \sqrt{\frac{1+v}{1-v}} = A \log. a^z$.

Erit $A \log. a^z = \frac{a^{2z}-1}{a^{2z}-1} + \frac{1}{3}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^3 + \frac{1}{5}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^5 + \frac{1}{7}\left(\frac{a^{2z}-1}{a^{2z}+1}\right)^7 + \ldots\ldots$

Sit $z = 1$; erit $A \log. a = \frac{a^2-1}{a^2+1} + \frac{1}{3}\left(\frac{a^2-1}{a^2+1}\right)^3 + \frac{1}{5}\left(\frac{a^2-1}{a^2+1}\right)^5 + \frac{1}{7}\left(\frac{a^2-1}{a^2+1}\right)^7 + \ldots$

Sit b baſis ſyſtematis logarithmici; unde $\log. a = \log. b = 1$;

$$A = \frac{bb-1}{bb+1} + \frac{1}{3}\left(\frac{bb-1}{bb+1}\right)^3 + \frac{1}{5}\left(\frac{bb-1}{bb+1}\right)^5 + \frac{1}{7}\left(\frac{bb-1}{bb+1}\right)^7 + \ldots\ldots$$

Obſervatio. Modulus cujusvis ſyſtematis eſt logarithmus naturalis baſeos hujus ſyſtematis. Etenim quoniam modulus logarithmorum naturalium ſeu hyperbolicorum eſt unitas, erit $A \log. b = \log. \text{hyp}. b$. Sed $\log. b = 1$; ergo $A = \log. \text{hyp}. b$.

Ex. gr. Baſis logarithmorum vulgarium ſeu Briggianorum eſt 10. Logarithmus hyperbolicus ipſius 10 eſt $2, 302\ 585^{0\,+}_{1\,-}$. Hinc modulus logarithmorum vulgarium eſt $2, 302\ 585^{0\,+}_{1\,-}$. Proinde ſi logarithmi hyperbolici per hunc numerum dividantur, oriuntur logarithmi vulgares; & viciſſim, ſi logarithmi vulgares per eundem numerum multiplicentur, oriuntur logarithmi hyperbolici. Cum autem logarithmorum hyperbolicorum uſus ſit quam frequentiſſimus; optandum ſane foret, ut tabulæ horum logarithmorum in promtu eſſent non minus extenſæ quam tabulæ vulgares.

§. 63. Poſtquam logarithmorum theoriam ex contemplatione progreſſionum geometricarum primariisque earundem proprietatibus deduxi, atque ita ad calculum elementarem revocavi: non inutile cenſeo in gratiam tironum idem ſubjectum paulo aliter tractare; ſtrictim ea exponendo, quæ in opuſculo *Expoſition elementaire* &c. de curva logarithmica fuſius tradidi.

Definitio. Curva logarithmica ſeu logiſtica ea eſt, in qua, abſciſſis in progreſſione arithmetica ſumptis, rectæ axi ordinatim applicatæ geometricam ſequuntur progreſſionem..

Corollarium. Ex definitione immediate ſequitur: partem axis abſciſſarum, inter duas rectas axi ordinatim applicatas comprehenſam, menſuram eſſe rationis harum ordinatarum; ita ut, differentia abſciſſarum manente eadem, ratio ordinatarum pariter eadem ſit; & viciſſim, ratione ordinatarum manente eadem, differentia abſciſſarum pariter eadem maneat. Si vero pars axis inter duas ordinatas ſit dupla, tripla, quadrupla, quintupla, - - - n^{tupla}; etiam ratio duarum ordinatarum eſt duplicata, triplicata, quadruplicata, quintuplicata, - - - n^{plicata}.

Quibus poſitis, ſint duæ ordinatæ, quarum ratio datur; & per puncta extrema harum ordinatarum ducatur recta, quæ axi occurrat, ſeu linea ſecans. Dico: ſubſecantem eſſe etiam datæ magnitudinis.

Fig. 16. Sint nimirum MP, $M'P'$ duæ rectæ axi ordinatim applicatæ. Ducatur MM', quæ axi occurrat in S. Sit ratio $MP : M'P'$ æqualis alicui rationi datæ; dico, lineam PS etiam eſſe magnitudine datam.

Ducatur $M'm$ axi parallela, quæ ordinatæ MP in m occurrat.

Quoniam ratio $MP : M'P'$ datur; ratio $MP : MP - M'P'$ ſeu $MP : Mm$ pariter datur. Sed $MP : Mm = PS : M'm$; ergo & ratio $PS : M'm$ datur. Sed propter rationem datam $MP : M'P'$, PP' ſeu $M'm$ datur magnitudine; ergo etiam PS datur magnitudine.

Atqui (§. 40.) ſubtangens cujusvis curvæ eſt limes ſubſecantis; ergo etiam, ſi a quocunque puncto logarithmicæ ducatur tangens, quæ axi occurrat, ſubtangens eſt datæ magnitudinis. Quæ propoſitio (ſi opus foret) immediate etiam poſſet demonſtrari, evincendo (per abſurdum) impoſſibilitatem inæqualitatis ſubtangentium diverſis logarithmicæ punctis reſpondentium.

Sit ideo $MP = y$, $AP = x$; & ſit t ſubtangens conſtans: erit $t = y\frac{\mathrm{d}x}{\mathrm{d}y}$, & $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{t}{y}$. Unde ſi $y = 1 + v$, fit

$$\frac{\mathrm{d}x}{\mathrm{d}y} = t \cdot \frac{1}{1+v} = t(1 - v + v^2 - v^3 + v^4 - v^5 + v^6 - \ldots)$$

$$\& \; x = t(v - \tfrac{1}{2}v^2 + \tfrac{1}{3}v^3 - \tfrac{1}{4}v^4 + \tfrac{1}{5}v^5 - \tfrac{1}{6}v^6 + \tfrac{1}{7}v^7 - \ldots)$$

Pro-

Proinde in diverſis curvis logarithmicis logarithmi ejusdem rationis ſunt in ratione ſubtangentium harum curvarum; & ſubtangentes inverſe ſunt ut moduli ſyſtematum logarithmicorum hisce curvis reſpondentium.

§. 64. Ex æquatione differentiali curvæ logarithmicæ $t = y\frac{dx}{dy}$; ſeu $\frac{dy}{dx} = \frac{y}{t}$ ſequitur

$$\frac{ddy}{dx^2} = \frac{\frac{dy}{dx}}{t} = \frac{y}{t^2}$$

$$\frac{d^3y}{dx^3} = \frac{\frac{dy}{dx}}{t^2} = \frac{y}{t^3}$$

$$\frac{d^4y}{dx^4} = \frac{\frac{dy}{dx}}{t^3} = \frac{y}{t^4}$$

$$\frac{d^5y}{dx^5} = \frac{\frac{dy}{dx}}{t^4} = \frac{y}{t^5}$$

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Fiat $x = x + \Delta x$

& proinde $y = y + \Delta y$.

Erit $y + \Delta y = y + \frac{\Delta x}{1}\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3} + \frac{\Delta x^4}{1.2...4}\frac{d^4y}{dx^4} + \ldots$ (§. 32.)

$$= y + \frac{\Delta x}{1}\frac{y}{t} + \frac{\Delta x^2}{1.2}\frac{y}{t^2} + \frac{\Delta x^3}{1.2.3}\frac{y}{t^3} + \frac{\Delta x^4}{1.2...4}\frac{y}{t^4} + \ldots$$

Atqui quoniam $y = e^x$; & $y + \Delta y = e^{x+\Delta x} = y \times e^{\Delta x}$,

erit $e^{\Delta x} = 1 + \frac{\Delta x}{1}\cdot\frac{1}{t} + \frac{\Delta x^2}{1.2}\cdot\frac{1}{t^2} + \frac{\Delta x^3}{1.2.3}\cdot\frac{1}{t^3} + \frac{\Delta x^4}{1.2...4}\cdot\frac{1}{t^4} + \frac{\Delta x^5}{1.2...5}\cdot\frac{1}{t^5} + \ldots$

Univerſim igitur

$$e^z = 1 + \frac{z}{1}\cdot\frac{1}{t} + \frac{z^2}{1.2}\cdot\frac{1}{t^2} + \frac{z^3}{1.2.3}\cdot\frac{1}{t^3} + \frac{z^4}{1.2...4}\cdot\frac{1}{t^4} + \frac{z^5}{1.2...5}\cdot\frac{1}{t^5} + \ldots$$

$$= 1 + \frac{1}{1}\cdot\frac{z}{t} + \frac{1}{1.2}\cdot\frac{z^2}{t^2} + \frac{1}{1.2.3}\cdot\frac{z^3}{t^3} + \frac{1}{1.2...4}\cdot\frac{z^4}{t^4} + \frac{1}{1.2...5}\cdot\frac{z^5}{t^5} + \ldots$$ (ut prius §. 55.)

§. 65. Quemadmodum curva logarithmica apta fuit ad theoriam logarithmorum illuſtrandam; ita & aliæ fingi poſſunt curvæ, quæ eidem ſcopo reſponderent. Celebris ob inſignes ſuas proprietates apud Geometras eſt curva *ſpiralis logarithmica* dicta. In hac curva angulus inter duos radios vectores comprehenſus eſt menſura rationis horum radiorum; unde ſequitur, data duorum radiorum ratione, ſpecie etiam dari triangulum his radiis vectoribus & chorda ipſorum extrema jungente comprehenſum. Proinde data ratione duorum radiorum; ſecans ſpiralis logarithmicæ, per puncta extrema horum radiorum ducta, inclinatur ſub angulo dato alterutri horum radiorum; unde concluditur (per §. 4.) angulum, quem tangens ſpiralis logarithmicæ facit cum radio ad punctum contactus ducto, conſtantem eſſe in eadem ſpirali logarithmica. Et diverſitas hujus anguli determinat ſyſtema logarithmicum ſpirali huic reſpondens. In hyperbola conica ad aſymptotos relata, ſpatia hyperbolica inter duas ordinatas comprehenſa creſcunt etiam ut logarithmi rationum abſciſſarum. (Vide *Caput X.*)

§. 66. Ex præcedentibus determinantur rationes differentiales quarumlibet quantitatum exponentialium, quod nonnullis exemplis oſtendam.

1°. Sit $P = x^m e^x$; erit

$$\frac{dP}{dx} = e^x(mx^{m-1} + x^m)$$

$$\frac{ddP}{dx^2} = e^x(m.m-1x^{m-2} + 2mx^{m-1} + x^m)$$

$$\frac{d^3P}{dx^3} = e^x(m.m-1.m-2x^{m-3} + 3m.m-1x^{m-2} + 3m.x^{m-1} + x^m)$$

$$\frac{d^4P}{dx^4} = e^x(m\ldots m-3x^{m-4} + 4m\ldots m-2x^{m-3} + 6m.m-1x^{m-2} + 4mx^{m-1} + x^m)$$

$$\frac{d^5P}{dx^6} = e^x(m\ldots m-4x^{m-5} + 5m\ldots m-3x^{m-4} + 10m\ldots m-2x^{m-3} + 10m.m-1x^{m-2} + 5mx^{m-1} + x^m)$$

$$\vdots$$

$$\frac{d^nP}{dx^n} = e^x(m..m-(n-1)x^{m-n} + \frac{n}{1}.m..m-(n-2)x^{m-n-1} + \frac{n}{1}.\frac{n-1}{2}m..m-(n-3)x^{m-n-2} + \ldots\frac{n}{1}mx^{m-1} + x^m)$$

2°. Sit

2°. Sit $P = x^y$. $\text{Log.}\,P = y\log.x$

$$\frac{dP}{dx} = P\frac{dy}{dx}\log.x + Py\frac{1}{x} = x^y\frac{dy}{dx}\log.x + yx^{y-1}.$$

3°. Sit $P = \log.^m x$;

$$\frac{dP}{dx} = m\log.^{m-1}x.\frac{1}{x}$$

$$\frac{ddP}{dx^2} = m.\overline{m-1}\log.^{m-2}x.\frac{1}{xx} - m\log.^{m-1}x.\frac{1}{xx}$$

$$\frac{d^3P}{dx^4} = m.\overline{m-1}.\overline{m-2}\log.^{m-3}x.\frac{1}{x^3} - 3m.\overline{m-1}\log.^{m-2}x.\frac{1}{x^3} + 2m.\log.^{m-1}x.\frac{1}{x^3}$$

&c. &c. &c.

§. 67. Suppoſita formula differentiali $\frac{d\log.x}{dx} = \frac{1}{x}$: ex illa deduci poſſent aliæ formulæ differentiales quantitatum non exponentialium.

Exempla.

1°. Sit $P = xy$

$$\log.P = \log.x + \log.y$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} + \frac{dy}{dx}.\frac{1}{y}$$

$$\frac{dP}{dx} = y + x\frac{dy}{dx}.$$

2°. Sit $P = xyzv\ldots.$

$$\log.P = \log.x + \log.y + \log.z + \log.v$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} + \frac{dy}{dx}.\frac{1}{y} + \frac{dz}{dx}.\frac{1}{z} + \frac{dv}{dx}.\frac{1}{v}.$$

$$\frac{dP}{dx} = yzv + \frac{dy}{dx}.xzv + \frac{dz}{dx}.xyv + \frac{dv}{dx}xyz.$$

3°. Sit $P = \frac{x}{y}$.

$$\log.P = \log.x - \log.y$$

$$\frac{dP}{dx}.\frac{1}{P} = \frac{1}{x} - \frac{dy}{dx}.\frac{1}{y}$$

$$\frac{dP}{dx} = \frac{1}{y} - x\frac{dy}{dx}.\frac{1}{yy} = \frac{y - x\frac{dy}{dx}}{yy}.$$

4°. Sit

4°. Sit $P = (a+x)^m$

$$\log. P = m\log.(a+x)$$

$$\frac{dP}{dx}.\frac{1}{P} = m.\frac{1}{a+x}$$

$$\frac{dP}{dx} = m.\frac{P}{a+x} = m(a+x)^{m-1}.$$

§. 68. Ad formulas differentiales, capite hoc traditas, reducuntur plurimæ aliæ formulæ differentiales; quæ proinde integrabiles cenſentur, tabulis logarithmicis in promtu habitis: quod paucis exemplis oſtendam.

Sit $\frac{dy}{dx} = \sqrt{(xx-aa)}$; fiat $xx-aa=(z-x)^2$; hinc $x=\frac{zz+aa}{2z}$; $z-x=\frac{zz-aa}{2z}$;

$\frac{dx}{dz}=\frac{1}{2}\left(\frac{zz-aa}{2z}\right)$; hinc $\frac{dy}{dz}=\frac{1}{4}\frac{(zz-aa)^2}{z^3}=\frac{1}{4}z-\frac{1}{2}aa\frac{1}{z}+\frac{1}{4}\frac{a^4}{z^3}$;

unde $y = \frac{1}{8}zz-\frac{1}{2}aa\log.z-\frac{1}{8}\frac{a^4}{zz}+C$

$$= \frac{1}{8}(x+\sqrt{(xx-aa)})^2-\frac{1}{2}aa\log.(x+\sqrt{xx-aa})-\frac{1}{8}\frac{a^4}{(x+\sqrt{(xx-aa)})^2}+C$$

$$= \frac{1}{8}(x+\sqrt{(xx-aa)})^2-\frac{1}{2}aa\log.(x+\sqrt{xx-aa})-\frac{1}{8}(x-\sqrt{xx-aa})^2+C$$

$$= \frac{1}{2}x\sqrt{(xx-aa)} \quad -\frac{1}{2}aa\log.(x+\sqrt{(xx-aa)})+C.$$

Si exponens integralis evaneſcat facto $x=a$, erit $C=+\frac{1}{2}aa\log.a$

$$\& \quad y = \frac{1}{2}x\sqrt{((xx-aa))}-\frac{1}{2}aa\log.\frac{x+\sqrt{(xx-aa)}}{a}$$

$$= \frac{1}{2}x\sqrt{(xx-aa)}-\frac{1}{2}aa\log.\frac{a}{x-\sqrt{(xx-aa)}}.$$

Sit $\frac{dy}{dx} = \sqrt{(xx+aa)}$;

fit eodem modo; $y = \frac{1}{2}x\sqrt{(xx+aa)}-\frac{1}{2}aa\log.\frac{a}{\sqrt{(xx+aa)}}$.

Ad has formulas reducuntur aliæ plurimæ, quales ſunt $\frac{dy}{dx} = x^{2m}\sqrt{(xx\pm aa)}$, quibus m eſt numerus integer.

Formulæ autem $\frac{dy}{dx} = x^{2m+1}\sqrt{(xx\pm aa)}$ integratio obtineri poteſt immediate; neque pendet a logarithmis.

§. 69.

§. 69. Formulæ differentiales logarithmicæ poſſunt etiam quantitatum in ſeries converſionem reddere faciliorem.

Exempla. Ponamus legem, quam coëfficientes poteſtatum binomii ſequuntur, nobis eſſe adhuc ignotam; & ſit

$$(1+x)^m = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \ldots$$

Sumendis logarithmis erit

$$m \log. 1+x = \log.(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \ldots$$

Sumendis exponentibus differentialibus erit

$$m \cdot \frac{1}{1+x} = \frac{A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + 6Fx^7 + 7Gx^6 + \ldots}{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \ldots}$$

$$\text{et } m(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \ldots)$$
$$= (1+x)(A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + 6Fx^5 + 7Gx^6 + \ldots)$$

Unde æquando coëfficientes poteſtatum ſimilium juxta methodum reverſionis ſerierum fit

$$A = m \qquad \text{unde} \qquad A = \frac{m}{1}$$
$$2B + A = mA \qquad B = \frac{m}{1} \cdot \frac{m-1}{2}$$
$$3C + 2B = mB \qquad C = \frac{m}{1} \cdot\cdot \frac{m-2}{3}$$
$$4D + 3C = mC \qquad D = \frac{m}{1} \cdots \frac{m-3}{4}$$
$$5E + 4D = mD \qquad E = \frac{m}{1} \cdots\cdot \frac{m-4}{5}$$
$$6F + 5E = mE \qquad F = \frac{m}{1} \cdots\cdots \frac{m-5}{6}$$
$$7G + 6F = mF \qquad G = \frac{m}{1} \cdots\cdots \frac{m-6}{7}$$

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2°. Eodem modo fit $(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots)^m$
$$= 1 + A'x + B'x^2 + C'x^3 + D'x^4 + E'x^5 + \ldots$$

Erit

$$\text{Erit} \quad m\log.(1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots$$
$$= \log.(1 + A'x + B'x^2 + C'x^3 + D'x^4 + E'x^5 + \ldots$$

$$\text{atque} \quad m \times \frac{A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \ldots}{1 + Ax + Bx^2 + Cx^3 + Dx^4 + \ldots}$$
$$= \frac{A' + 2B'x + 3C'x^2 + 4D'x^3 + 5E'x^4 + \ldots}{1 + A'x + B'x^2 + C'x^3 + D'x^4 + \ldots}$$

Unde coëfficientes A', B', C', D', E' .... facilius quam aliis methodis determinari poſſunt

3°. Sit $a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots$

$$x\log.a = \log.\, 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots$$
$$\log.a = \frac{A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \ldots}{1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots}$$
$$\log.a\, (1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots)$$
$$= A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \ldots$$

Unde $A = \log.a$ ſeu $A = \log.a$

$B = A\frac{\log.a}{2}$ $\qquad B = \frac{1}{1.2}\log.^2 a$

$C = B\frac{\log.a}{3}$ $\qquad C = \frac{1}{1.2.3}\log.^3 a$

$D = C\frac{\log.a}{4}$ $\qquad D = \frac{1}{1.2\ldots4}\log.^4 a$

$E = D\frac{\log.a}{5}$ $\qquad E = \frac{1}{1.2\ldots5}\log.^5 a$

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$$a^x = 1 + x\log.a + \frac{x^2}{1.2}\log.^2 a + \frac{x^3}{1.2..3}\log.^3 a + \frac{x^4}{1.2..4}\log.^4 a + \frac{x^5}{1.2..5}\log.^5 a + \ldots\ldots$$

(ut prius §. 55.)

4°. Sit $\log.\, 1 + x = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \ldots$

$$\text{erit} \quad \frac{1}{1+x} = A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \ldots$$
$$1 = A + x\begin{matrix}A\\+2B\end{matrix} + x^2\begin{matrix}2B\\+3C\end{matrix} + x^3\begin{matrix}3C\\+4D\end{matrix} + x^4\begin{matrix}4D\\+5E\end{matrix} + \ldots$$

Unde

Unde $A = + 1$

$B = - \frac{1}{2}$

$C = + \frac{1}{3}$

$D = - \frac{1}{4}$

$E = + \frac{1}{5}.$

Et $\log. 1 + x = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$ (ut prius §. 61.)

§. 70. Dixi (§. 78.), problema inverſum tangentium ad formulas differentiales logarithmicas ſæpe etiam in caſibus ſimpliciſſimis reduci.

Exempla. 1°. Quæratur curva, cujus ſubtangens datam habet ad abſciſſam rationem.

Erit ideo $y\frac{dx}{dy} = mx.$

hinc $\frac{dy}{dx}.\frac{1}{y} = \frac{1}{mx} = \frac{1}{m}.\frac{1}{x}$

ſeu $\frac{dy}{dv}.\frac{1}{y} = \frac{1}{m}\frac{dx}{dv}.\frac{1}{x}$

hinc $\log.\frac{y}{v} = \frac{1}{m}\log.\frac{x}{v} + C$

$m\log. y = \log. x + C$

$y^m = Cx$, quæ eſt æquatio ad parabolam, ſi m ſit quantitas poſitiva ab unitate diverſa; ad lineam rectam, ſi $m = 1$; & ad hyperbolas, ſi m fuerit negativa, ſeu ſi ſubtangens & abſciſſa ad diverſas ordinatæ partes ſitæ ſint.

2°. Quæritur curva, cujus ſubtangens eſt conſtans.

Erit ideo $y\frac{dx}{dy} = c$; $\frac{dy}{dx}.\frac{1}{y} = \frac{1}{c}$; $\log. y = \frac{x}{c}$, quæ eſt æquatio ad logarithmicam.

3°. Quæritur curva, cujus tangens datur magnitudine.

Sit ideo $y\surd(1 + \frac{dx^2}{dy^2}) = t$; $1 + \frac{dx^2}{dy^2} = \frac{tt}{yy}$; $\frac{dx^2}{dy^2} = \frac{tt-yy}{yy}.$

$$\frac{dx}{dy} = \frac{\surd(tt-yy)}{y} = \frac{tt}{y\surd(tt-yy)} - \frac{y}{\surd(tt-yy)};$$

$x = C + t\log.\frac{y}{t+\surd(tt-yy)} + \surd(tt-yy)$, quæ eſt æquatio ad curvam *tractoriam* dictam.

O

§. 71.

§. 71. Priuſquam huic capiti coronidem imponam; pauciſſimis attingam quæſtionem inter mathematicos nimium celebrem, de natura logarithmorum quantitatum negativarum: quæ quæſtio nequidem mota fuiſſet, ſi algebriſtæ a legitima quantitatum homogenearum definitione Euclidea non receſſiſſent. Scilicet: *rationem inter ſe magnitudines habere dicuntur, quæ multiplicatæ ſe invicem ſuperare poſſunt.* Jam vero quantitas (ita dicta) negativa, quotiescunque multiplicetur (hoc eſt juxta genuinam multiplicationis notionem, quotiescunque ſibi ipſi addatur, ſeu repetatur), nunquam ſuperabit quantitatem poſitivam. Ergo nulla datur ratio inter quantitates (ſic dictas) poſitivas & negativas.

Nodus autem ſolutionis hic eſt. Algebriſtæ ad quantitates ipſas tranſtulerunt, quæ ad ſigna tantum pertinebant: omnes quantitates in ſe ſpectatæ ſunt poſitivæ, ſed tantum aliæ aliis ſunt oppoſitæ: hanc oppoſitionem per ſigna +, −, deſignatam non licebat ad quantitates ipſas transferre, & quaſi duos ordines quantitatum hoc reſpectu conſtituere. Quæſtionem hanc lucide admodum enodavit, & ad genuina principia revocavit illuſtris Kæstnerus (*Leipziger Magazin für die reine und angewandte Mathematik*, 1786. *4tes Stück*): qui (contra opinionem tum Joh. Bernoulli, tum & D. d'Alembert) Leibnitzii & Euleri ſententiæ aſſentitur; nempe, logarithmos quantitatum (ita dictarum) negativarum eſſe imaginarios; ſeu, nullam eſſe rationem inter negativas (quas vocant) quantitates & poſitivas. Quod attinet ad computum horum logarithmorum imaginariorum, vid. Euler *Memoires de Berlin* 1749.

Caput Septimum.

De functionibus trigonometricis arcuum circularium.

§. 72.

Analogia, quæ inter quantitates exponentiales a^z, & functiones trigonometricas arcuum circularium, quales ſunt ſinus, coſinus, tangentes &c. intercedit, jam dudum fuit a mathematicis obſervata. Fundamenta autem analogiæ hujus methodo ſatis elementari & lucida (quod ſciam) expoſita non fuerunt; qua id me præſtitiſſe arbitror, ſinus & coſinus cujusvis arcus circularis expreſſio-

preſſionem per hunc arcum iisdem omnino veſtigiis indagando, quibus in præcedente capite inſiſtebam.

Cum ſin. z ejusmodi ſit ipſius z functio, quæ evaneſcit poſito $z=0$; & ratio æqualitatis limes ſit rationis decreſcentis, quæ arcum inter & ſinum ejus intercedit: ſinus cujusvis arcus z eſt functio ipſius z, talis - - - -

$$\text{ſin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + Ez^6 + \ldots$$

Et quoniam coſ. z eſt functio ipſius z talis, ut fiat 1, quando $z=0$; minor autem ſit unitate, quando z creſcit ab zero inde usque ad quadrantem: erit coſ. z functio ipſius z talis - - - -

$$\text{coſ.}\, z = 1 - A'z + B'z^2 + C'z^3 + D'z^4 + E'z^5 + \ldots$$

§. 73. Sit itaque

$$\text{ſin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + Ez^6 + \ldots$$

erit

$$\text{ſin.}\, 2z = 2z - 2^2Az^2 + 2^3Bz^3 + 2^4Cz^4 + 2^5Dz^5 + 2^6Ez^6 + \ldots$$

$$\text{ſin.}\, 3z = 3z - 3^2Az^2 + 3^3Bz^3 + 3^4Cz^4 + 3^5Dz^5 + 3^6Ez^6 + \ldots$$

$$\text{ſin.}\, 4z = 4z - 4^2Az^2 + 4^3Bz^3 + 4^4Cz^4 + 4^5Dz^5 + 4^6Ez^6 + \ldots$$

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Sumtis differentiis primis erit (§. *t. Introd.*)

$$2\,\text{ſin.}\tfrac{1}{2}z\,\text{coſ.}\tfrac{3}{2}z = z - (2^2-1)Az^2 + (2^3-1)Bz^3 + (2^4-1)Cz^4 + (2^5-1)Dz^5 + (2^6-1)Ez^6 + \ldots$$

$$2\,\text{ſin.}\tfrac{1}{2}z\,\text{coſ.}\tfrac{5}{2}z = z - (3^2-2^2)Az^2 + (3^3-2^3)Bz^3 + (3^4-2^4)Cz^4 + (3^5-2^5)Dz^5 + (3^6-2^6)Ez^6 + \ldots$$

$$2\,\text{ſin.}\tfrac{1}{2}z\,\text{coſ.}\tfrac{7}{2}z = z - (4^2-3^2)Az^2 + (4^3-3^3)Bz^3 + (4^4-3^4)Cz^4 + (4^5-3^5)Dz^5 + (4^6-3^6)Ez^6 + \ldots$$

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Jam vero in his æquationibus primus terminus omnium membrorum priorum, ſi juxta ſuppoſitionem §. 72. in ſeries explicentur, eſt $1z$; pariter atque primus terminus omnium membrorum poſteriorum, qui igitur per ſe congruunt,

Sumtis differentiis ſecundis erit (§. *t. Introd.*)

$$-2^2\,\text{ſin.}^2\tfrac{1}{2}z\,\text{ſin.}\,2z = -\Delta''(3^2\ldots1^2)Az^2 + \Delta''(3^3\ldots1^3)Bz^3 + \Delta''(3^4\ldots1^4)Cz^4 + \ldots$$

$$-2^2\,\text{ſin.}^2\tfrac{1}{2}z\,\text{ſin.}\,3z = -\Delta''(4^2\ldots2^2)Az^2 + \Delta''(4^3\ldots2^3)Bz^3 + \Delta''(4^4\ldots2^4)Cz^4 + \ldots$$

$$-2^2\,\text{ſin.}^2\tfrac{1}{2}z\,\text{ſin.}\,4z = -\Delta''(5^2\ldots3^2)Az^2 + \Delta''(5^3\ldots3^3)Bz^3 + \Delta''(5^4\ldots3^4)Cz^4 + \ldots$$

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Atqui

Atqui primus terminus omnium membrorum priorum, ſi juxta §. 72. in ſeries explicentur, continet tertiam poteſtatem z^3 arcus mutabilis z; ita ut membra hæc evoluta ſecundam arcus hujus poteſtatem z^2 non contineant: proinde poſteriora etiam æquationum harum membra poteſtatem z^2 non continebunt. Sed in poſterioribus membris coëfficiens poteſtatis ſecundæ z^2 eſt $\Delta'' n^2 A = 1.2A$: proinde $1.2A = 0$, & $A = 0$.

Eodem modo demonſtratur in ſerie $\text{ſin.}\, z = z - Az^2 + Bz^3 + Cz^4 + Dz^5 + \ldots$ omnes deeſſe arcus mutabilis poteſtates exponentis paris. Scilicet: ſumtis differentiis ordinis $2m$, quæ ſunt $\pm\, 2^{2m}\text{ſin.}^{2m}\tfrac{1}{2}z\,\text{ſin.}\,pz$ (§. t. *Introd.*), primus terminus omnium membrorum priorum continet poteſtatem z^{2m+1}; ita ut in his membris deſit poteſtas par z^{2m}.

At vero primi termini omnium membrorum poſteriorum continent hanc poteſtatem parem, affectam coëfficiente, cujus unus factor eſt conſtans $\Delta^{2M} n^{2m} = 1.2.3\ldots 2m$ (§. q. *Introd.*). Quare cum potentia hæc exponentis paris ex ſecundis etiam membris debeat exulare, alter coëfficientis illius factor evaneſcat oportet; & proinde ſeries, qua ſinus per arcum exprimitur, hujus eſt formæ $\text{ſin.}\, z = z - Az^3 + Bz^5 + Cz^7 + Dz^9 + \ldots$

Sumtis differentiis tertiis erit (§.)

$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{5}{2}z = \Delta'''(4^3\ldots1^3)Az^3 - \Delta'''(4^5\ldots1^5)Bz^5 - \Delta'''(4^7\ldots1^7)Cz^7 - \ldots$$
$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{7}{2}z = \Delta'''(5^3\ldots2^3)Az^3 - \Delta'''(5^5\ldots2^5)Bz^5 - \Delta'''(5^7\ldots2^7)Cz^7 - \ldots$$
$$2^3\text{ſin.}^3\tfrac{1}{2}z\,\text{coſ.}\tfrac{9}{2}z = \Delta'''(6^3\ldots3^3)Az^3 - \Delta'''(6^5\ldots3^5)Bz^5 - \Delta'''(6^7\ldots3^7)Cz^7 - \ldots$$

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Quoniam primus terminus omnium membrorum priorum juxta §. 72. in ſeries evolutorum eſt $1z^3$, erit etiam primus terminus omnium membrorum poſteriorum $1z^3$. Sed (§. q. *Introd.*) $\Delta''' n^3 = 1.2.3$; ergo $1 = 1.2.3A$, & $A = \frac{1}{1.2.3}$.

Sumtis differentiis quartis & deinceps quintis erit

$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{7}{2}z = \Delta^{v}(6^5\ldots1^5)Bz^5 + \Delta^{v}(6^7\ldots1^7)Cz^7 + \Delta^{v}(6^9\ldots1^9)Dz^9 + \ldots$$
$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{9}{2}z = \Delta^{v}(7^5\ldots2^5)Bz^5 + \Delta^{v}(7^7\ldots2^7)Cz^7 + \Delta^{v}(5^9\ldots2^9)Dz^9 + \ldots$$
$$2^5\text{ſin.}^5\tfrac{1}{2}z\,\text{coſ.}\tfrac{11}{2}z = \Delta^{v}(8^5\ldots3^5)Bz^5 + \Delta^{v}(8^7\ldots3^7)Cz^7 + \Delta^{v}(8^9\ldots3^9)Dz^9 + \ldots$$

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Primus

Primus terminus omnium membrorum priorum eſt $1z^5$ (§. 72.); ergo & primus terminus omnium membrorum poſteriorum eſt $1z^5$. Sed $\Delta^{v} n^5 = 1.2\ldots5$ (§. q. *Introd.*); ergo $1.2\ldots5B = 1$, & $B = \frac{1}{1.2\ldots5}$.

Sumtis differentiis ſextis & deinceps ſeptimis erit

$$-2^7 \text{ſin.}^7 \tfrac{1}{2}z \text{ coſ.} \tfrac{9}{2}z = \Delta^{\text{VII}}(8^7\ldots1^7)Cz^7 + \Delta^{\text{VII}}(8^9\ldots1^9)Dz^9 + \Delta^{\text{VII}}(8^{11}\ldots1^{11})Ez^{11} + \ldots$$

$$-2^7 \text{ſin.}^7 \tfrac{1}{2}z \text{ coſ.} \tfrac{11}{2}z = \Delta^{\text{VII}}(9^7\ldots2^7)Cz^7 + \Delta^{\text{VII}}(9^5\ldots2^5)Dz^9 + \Delta^{\text{VII}}(9^{11}\ldots2^{11})Ez^{11} + \ldots$$

$$-2^7 \text{ſin.}^7 \tfrac{1}{2}z \text{ coſ.} \tfrac{13}{2}z = \Delta^{\text{VII}}(10^7\ldots3^7)Cz^7 + \Delta^{\text{VII}}(10^9\ldots3^9)Dz^9 + \Delta^{\text{VII}}(10^{11}\ldots3^{11})Ez^{11} + \ldots$$

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Primus terminus omnium priorum membrorum evolutorum eſt $-1z^7$; ergo & primus terminus omnium membrorum poſteriorum eſt $-1z^7$. Sed hic primus terminus eſt $\Delta^{\text{VII}} n^7 Cz^7 = 1.2\ldots7Cz^7$; proinde $-1 = 1.2\ldots7C$, & $C = -\frac{1}{1.2\ldots7}$.

Eodem modo ſumtis differentiis octavis, & deinceps nonis, erit

$1z^9 = \Delta^{\text{IX}} n^9 Dz^9 = 1.2\ldots9Dz^9$; unde $D = \frac{1}{1.2\ldots9}$;

pariter $E = -\frac{1}{1.2\ldots11}$

$F = \frac{1}{1.2\ldots13}$; & generatim

$$\text{ſin.}\, z = z - \frac{1}{1.2.3}z^3 + \frac{1}{1.2\ldots5}z^5 - \frac{1}{1.2\ldots7}z^7 + \frac{1}{1.2\ldots9}z^9 - \frac{1}{1.2\ldots11}z^{11} + \ldots.$$

§. 74. Inveſtigatio coſinus eodem modo peragitur.

Sit nempe $\text{coſ.}\, z = 1 - Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \ldots.$

hinc $\text{coſ.}\, 2z = 1 - 2Az + 2^2Bz^2 + 2^3Cz^3 + 2^4Dz^4 + 2^5Ez^5 + \ldots.$

$\text{coſ.}\, 3z = 1 - 3Az + 3^2Bz^2 + 3^3Cz^3 + 3^4Dz^4 + 3^5Ez^5 + \ldots.$

$\text{coſ.}\, 4z = 1 - 4Az + 4^2Bz^2 + 4^3Cz^3 + 4^4Dz^4 + 4^5Ez^5 + \ldots.$

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Hinc ſumtis differentiis primis erit

$$2\text{ſin.}\tfrac{1}{2}z\, \text{ſin.}\tfrac{3}{2}z = Az - (2^2-1)Bz^2 - (2^3-1)Cz^3 - (2^4-1)Dz^4 - (2^5-1)Ez^5 - \ldots.$$

$$2\text{ſin.}\tfrac{1}{2}z\, \text{ſin.}\tfrac{5}{2}z = Az - (3^2-2^2)Bz^2 - (3^3-2^3)Cz^3 - (3^4-2^4)Dz^4 - (3^5-2^5)Ez^5 - \ldots.$$

$$2\text{ſin.}\tfrac{1}{2}z\, \text{ſin.}\tfrac{7}{2}z = Az - (4^2-3^2)Bz^2 - (4^3-3^3)Cz^3 - (4^4-3^4)Dz^4 - (4^5-3^5)Ez^5 - \ldots.$$

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Quoniam autem (§. 72.) primus terminus omnium priorum membrorum non continet primam poteſtatem z arcus mutabilis z, prima hæc poteſtas deerit etiam in poſterioribus membris; unde erit $A = 0$.

Eodem autem modo, quo prius pro ſinu factum fuit, demonſtratur omnes poteſtates impares quantitatis mutabilis z evaneſcere. Scilicet differentiæ ordinis imparis $2m+1$ membrorum priorum ſunt $2^{2m+1}\,\text{ſin.}^{2m+1}\tfrac{1}{2}z\,\text{ſin.}pz$; & proinde primi termini horum membrorum evolutorum continent poteſtatem parem z^{2m+2} arcus mutabilis z: membra ergo priora harum æquationum non continent poteſtatem imparem z^{2m+1} arcus z; unde & membris poſterioribus poteſtates hæ impares debent exulare. Sed primi termini horum membrorum ſunt $\Delta^{2M+1}n^{2m+1}Mz^{2m+1}$, ſeu $1.2\ldots 2m+1\,Mz^{2m+1}$; proinde $0 = 1.2\ldots 2m+1\,M$, & $M = 0$. Coſinus itaque per arcum exprimitur ſerie hujus formæ, quæ pares tantum poteſtates ipſius z continet.

$$\text{coſ.}\,z = 1 - Az^2 + Bz^4 + Cz^6 + Dz^8 + Ez^{10} + \ldots.$$

Sumtis differentiis ſecundis erit

$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}2z = \Delta''(3^2\ldots1^2)Az^2 - \Delta''(3^4\ldots1^4)Bz^4 - \Delta''(3^6\ldots1^6)Cz^6 - \ldots$$
$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}3z = \Delta''(4^2\ldots2^2)Az^2 - \Delta''(4^4\ldots2^4)Bz^4 - \Delta''(4^6\ldots2^6)Cz^6 - \ldots$$
$$2^2\text{ſin.}^2\tfrac{1}{2}z\,\text{coſ.}4z = \Delta''(5^2\ldots3^2)Az^2 - \Delta''(5^4\ldots3^4)Bz^4 - \Delta''(5^6\ldots3^6)Cz^6 - \ldots$$

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Et quoniam primus terminus omnium priorum membrorum evolutorum eſt $1z^2$, erit etiam primus terminus omnium poſteriorum membrorum $1z^2$. Sed $\Delta''n^2 = 1.2$ (§. q. *Introd.*); ergo $1.2A = 1$, & $A = \frac{1}{1.2}$.

Tum ſumtis differentiis tertiis & deinceps quartis, erit

$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}3z = \Delta^{IV}(5^4\ldots1^4)Bz^4 + \Delta^{IV}(5^6\ldots1^6)Cz^6 + \Delta^{IV}(5^8\ldots1^8)Dz^8 + \ldots$$
$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}4z = \Delta^{IV}(6^4\ldots2^4)Bz^4 + \Delta^{IV}(6^6\ldots2^6)Cz^6 + \Delta^{IV}(6^8\ldots2^8)Dz^8 + \ldots$$
$$2^4\text{ſin.}^4\tfrac{1}{2}z\,\text{coſ.}5z = \Delta^{IV}(7^4\ldots3^4)Bz^4 + \Delta^{IV}(7^6\ldots3^6)Cz^6 + \Delta^{IV}(7^8\ldots3^8)Dz^8 + \ldots$$

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Unde erit $1z^4 = \Delta^{IV}n^4Bz^4 = 1.2\ldots4Bz^4$; & $B = \frac{1}{1.2\ldots4}$.

Tum

Tum ſumtis differentiis quintis & deinceps ſextis, erit

$$-2^6\,\text{ſin.}^6\tfrac{1}{2}z\,\text{coſ.}4z = \Delta^{VI}(7^6\ldots1^6)Cz^6+\Delta^{VI}(7^8\ldots1^8)Dz^8+\Delta^{VI}(7^{10}\ldots1^{10})Ez^{10}+\ldots$$

$$-2^6\,\text{ſin.}^6\tfrac{1}{2}z\,\text{coſ.}5z = \Delta^{IV}(8^6\ldots2^6)Cz^6+\Delta^{VI}(8^8\ldots2^8)Dz^8+\Delta^{VI}(8^{10}\ldots2^{10})Ez^{10}+\ldots$$

$$-2^6\,\text{ſin.}^6\tfrac{1}{2}z\,\text{coſ.}6z = \Delta^{IV}(9^6\ldots3^6)Cz^6+\Delta^{VI}(9^8\ldots3^8)Dz^8+\Delta^{VI}(9^{10}\ldots3^{10})Ez^{10}+\ldots$$

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Unde erit $-1z^6=\Delta^{VI}n^6Cz^6=1.2\ldots6Cz^6$: $C=-\frac{1}{1.2\ldots6}$.

Sumtis differentiis ſeptimis & deinceps octavis, erit pariter $1z^8=\Delta^{VIII}n^8Dz^8$, & $D=\frac{1}{1.2\ldots8}$.

Sumtis differentiis nonis & deinceps decimis, erit $-1z^{10}=\Delta^{X}n^{10}Ez^{10}$; & $E=-\frac{1}{1.2\ldots10}$. Unde tandem

$$\text{coſ.}z = 1-\frac{1}{1.2}z^2+\frac{1}{1.2\ldots4}z^4-\frac{1}{1.2\ldots6}z^6+\frac{1}{1.2\ldots8}z^8-\frac{1}{1.2\ldots10}z^{10}+\ldots.$$

§. 75. *Corollarium.* $\text{Tang.}z=\dfrac{z-\frac{1}{1.2.3}z^3+\frac{1}{1.2..5}z^5-\frac{1}{1.2..7}z^7+\frac{1}{1.2..9}z^9+\ldots.}{1-\frac{1}{1.2}z^2+\frac{1}{1.2.4}z^4-\frac{1}{1.2..6}z^6+\frac{1}{1.2..8}z^8-\ldots.}$

Ad ſcopum præſentem non pertinet expreſſionem hanc fractam in ſeriem convertere.

Obſervare ſufficiat, tang. z exprimi etiam per arcum z ſerie hujus formæ $\text{tang.}z=z+Az^3+Bz^5+Cz^7+Dz^9+\ldots.$, in qua $A=\frac{1}{3}$; $B=\frac{2}{15}\ldots.$

§. 76. Quoniam $\text{ſin.}z=z-\frac{1}{1.2.3}z^3+\frac{1}{1.2..5}z^5-\frac{1}{1.2..7}z^7+\frac{1}{1.2..9}z^9-\ldots.$

$$\text{eſt}\quad \frac{d\,\text{ſin.}z}{dz}=1-\frac{1}{1.2}z^2+\frac{1}{1.2..4}z^4-\frac{1}{1.2..6}z^6+\frac{1}{1.2..8}z^8-\ldots=$$

$=\text{coſ.}z$ (conformiter §. 49.)

et quoniam $\text{coſ.}z=1-\frac{1}{1.2}z^2+\frac{1}{1.2..4}z^4-\frac{1}{1.2..6}z^6+\frac{1}{1.2..8}z^8-\ldots.$

$$\text{eſt}\quad \frac{d\,\text{coſ.}z}{dz}=-z+\frac{1}{1.2.3}z^3-\frac{1}{1.2..5}z^5+\frac{1}{1.2..7}z^7-\ldots=$$

$=-\text{ſin.}z$ (conformiter §. 49.)

Et

Et quoniam $\frac{\text{ſin.}\,z}{\text{coſ.}z} = \text{tang.}\,z$; $\frac{\text{coſ.}\,z\frac{d\,\text{ſin.}z}{dz} - \text{ſin.}z\frac{d\,\text{coſ.}z}{dz}}{\text{coſ.}^2 z} = \frac{d\,\text{tang.}\,z}{dz}$;

ſeu $\frac{\text{coſ.}^2 z + \text{ſin.}^2 z}{\text{coſ.}^2 z} = \frac{d\,\text{tang.}\,z}{dz}$;

unde $\frac{d\,\text{tang.}\,z}{dz} = \frac{1}{\text{coſ.}^2 z} = \text{ſec.}^2 z$,

ſeu $\frac{dt}{dz} = \text{ſec.}^2 z$; et $\frac{dz}{dt} = \frac{1}{1+tt}$, quod geometrice etiam ſic demonſtratur.

Fig. 17. Sit C centrum & CA radius circuli, cujus arcus mutabilis ſit AXX'. Sit ab A ducta tangens hujus arcus; & ducantur CX, CX', quæ tangenti occurrant in Y & Y': tum centro C & radio CY deſcribatur arcus Yy, occurrens ipſi CX' in y.

$$\begin{aligned} &\text{Erit} \quad \lim.YY' : Yy = 1 : \text{ſin.}Y' = 1 : \text{ſin.}CYA = CY : CA \\ &\qquad\qquad Yy : XX' = CY : CA \\ &\text{ergo} \quad \lim.YY' : XX' = CY^2 : CA^2 \quad (\S.\ 14.) \end{aligned}$$

Sed eſt $\lim.YY' : XX' = \frac{dt}{ds}$; ergo $\frac{dt}{ds} = \frac{rr+tt}{rr}$; et $\frac{ds}{dt} = \frac{rr}{rr+tt}$.

§. 77. Quoniam ſinus & coſinus functiones ſunt arcus, expreſſiones eorum potuiſſent per theorema Taylorianum obtineri modo ſequenti:

$$\text{ſin.}z \pm \Delta z = \text{ſin.}z \pm \frac{\Delta z}{1}\frac{d\,\text{ſin.}z}{dz} + \frac{\Delta z^2}{1.2}\frac{d^2\,\text{ſin.}z}{dz^2} \pm \frac{\Delta z^3}{1.2.3}\frac{d^3\,\text{ſin.}z}{dz^3} + \frac{\Delta z^4}{1.2..4}\frac{d^4\text{ſin.}z}{dz^4} \pm \ldots$$

Atqui eſt (§. 76.) $\frac{d\,\text{ſin.}z}{dz} = +\,\text{coſ.}z$, et $\frac{d\,\text{coſ.}z}{dz} = -\,\text{ſin.}z$

unde eſt
$$\frac{d^2\,\text{ſin.}z}{dz^2} = -\,\text{ſin.}z$$
$$\frac{d^3\text{ſin.}z}{dz^3} = -\,\text{coſ.}z$$
$$\frac{d^4\text{ſin.}z}{dz^4} = +\,\text{ſin.}z$$
$$\frac{d^5\text{ſin.}z}{dz^5} = +\,\text{coſ.}z$$
$$\frac{d^6\text{ſin.}z}{dz^6} = -\,\text{ſin.}z$$
$$\vdots \qquad \vdots$$

hinc

Hinc

$$\text{ſin.}z\pm\Delta z=\text{ſin.}z\pm\frac{\Delta z}{1}\text{coſ.}z-\frac{\Delta z^2}{1.2}\text{ſin.}z\mp\frac{\Delta z^3}{1.2.3}\text{coſ.}z+\frac{\Delta z^4}{1...4}\text{ſin.}z\pm\frac{\Delta z^5}{1..5}\text{coſ.}z-\frac{\Delta z^6}{1..6}\text{ſin.}z\mp..$$

Unde eſt $\frac{\text{ſin.}(z+\Delta z)+\text{ſin.}(z-\Delta z)}{2}=\text{ſin.}z-\frac{\Delta z^2}{1.2}\text{ſin.}z+\frac{\Delta z^4}{1...4}\text{ſin.}z-\frac{\Delta z^6}{1...6}\text{ſin.}z+...$

$$\frac{\text{ſin.}(z+\Delta)z-\text{ſin.}(z-\Delta z)}{2}=\frac{\Delta z}{1}\text{coſ.}z-\frac{\Delta z^3}{1..3}\text{coſ.}z+\frac{\Delta z^5}{1..5}\text{coſ.}z-\frac{\Delta z^4}{1..7}\text{coſ.}z+..$$

ſeu $\text{ſin.}z\,\text{coſ.}\Delta z=\text{ſin.}z-\frac{\Delta z^2}{1.2}\text{ſin.}z+\frac{\Delta z^4}{1..4}\text{ſin.}z-\frac{\Delta z^6}{1..6}\text{ſin.}z+\dots$

$$\text{coſ.}z\,\text{ſin.}\Delta z=\frac{\Delta z}{1}\text{coſ.}z-\frac{\Delta z^3}{1..3}\text{coſ.}z+\frac{\Delta z^5}{1..5}\text{coſ.}z-\frac{\Delta z^7}{1..7}\text{coſ.}z+...$$

unde $\text{coſ.}\Delta z=1-\frac{\Delta z^2}{1.2}+\frac{\Delta z^4}{1...4}-\frac{\Delta z^6}{1...6}+\dots.$

$$\text{ſin.}\Delta z=\frac{\Delta z}{1}-\frac{\Delta z^3}{1.2.3}+\frac{\Delta z^5}{1...5}-\frac{\Delta z^7}{1...7}+\dots.$$

Proinde generatim $\text{coſ.}z=1-\frac{z^2}{1.2}+\frac{z^4}{1.2...4}-\frac{z^6}{1.2...6}+\dots.$

$$\text{ſin.}z=z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2...5}-\frac{z^7}{1.2...7}+\dots.$$

§. 78. Vidimus (§. 59.), quod

$$\frac{e^z+e^{-z}}{2}=1+\frac{z^2}{1.2}+\frac{z^4}{1.2...4}+\frac{z^6}{1.2...6}+\frac{z^8}{1.2...8}+\dots.$$

$$\frac{e^z-e^{-z}}{2}=z+\frac{z^3}{1.2.3}+\frac{z^5}{1.2...5}+\frac{z^7}{1.2...7}+\frac{z^9}{1.2...9}+\dots.$$

Series hæ magnam reſpective habent affinitatem cum expreſſionibus coſ.z & ſin.z. Etenim ſi in priori ſerie pro $+z^2$ ſubſtituatur $-z^2$, ſeu ſi loco z ſubſtituatur $z\sqrt{-1}$; prodit $1-\frac{z^2}{1.2}+\frac{z^4}{1.2...4}-\frac{z^6}{1.2...6}+\frac{z^8}{1.2...8}-\dots.$ ſeu coſ.z: unde eadem ſubſtitutione priori æquationis membro applicata, infertur

$$\text{coſ.}z=\frac{e^{+z\sqrt{-1}}+e^{-z\sqrt{-1}}}{2}.$$

Pariter in ſerie poſteriori, loco z ſubſtituto ſigno $z\sqrt{-1}$, prodit
$\sqrt{-1}\,(z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2...5} - \frac{z^7}{1.2...7} + \frac{z^9}{1.2...9} - \ldots)$, ſeu $\sqrt{-1}\,\text{ſin.}\,z$; unde ſimiliter infertur $\sqrt{-1}\,\text{ſin.}\,z = \frac{e^{+z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2}$, ſeu $\text{ſin.}\,z = \frac{e^{+z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}$.

Hinc $\text{tang.}\,z = \frac{1}{\sqrt{-1}} \times \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}$, ſeu $\text{tang.}\,z\sqrt{-1} = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}$; unde
$1 + \text{tang.}\,z\sqrt{-1} : 1 - \text{tang.}\,z\sqrt{-1} = e^{z\sqrt{-1}} : e^{-z\sqrt{-1}} = e^{2z\sqrt{-1}} : 1$;
$e^{2z\sqrt{-1}} = \frac{1 + \text{tang.}\,z\sqrt{-1}}{1 - \text{tang.}\,z\sqrt{-1}}$; $2z\sqrt{-1} = \log.\frac{1 + \text{tang.}\,z\sqrt{-1}}{1 - \text{tang.}\,z\sqrt{-1}}$;
$z = \frac{1}{2\sqrt{-1}}\log.\frac{1 + \text{tang.}\,z\sqrt{-1}}{1 - \text{tang.}\,z\sqrt{-1}}$, qua formula continetur fundamentum calculi logarithmorum quantitatum (ita dictarum) tam negativarum quam imaginariarum (vid. §. 71.).

Obſervatio. Formulæ exponentiales imaginariæ $\text{coſ.}\,z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2}$,
$\text{ſin.}\,z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}$, $\text{tang.}\,z = \frac{1}{\sqrt{-1}}\,\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}$, $z = \frac{1}{2\sqrt{-1}}\log.\frac{1 + \text{tang.}\,z\sqrt{-1}}{1 - \text{tang.}\,z\sqrt{-1}}$, frequentiſſime in calculo integrali uſurpantur, & plurimas inveſtigationes mirifice juvant. Spectari autem debent tanquam mera ſigna majoris facilitatis cauſa a mathematicis introducta; atque irrita foret inveſtigatio ſignificationis ſymbolorum $e^{z\sqrt{-1}}$, $e^{-z\sqrt{-1}}$, quibus ſeorſim ſumtis nulla idea reſpondet; nec operationes in ipſis inſtitutæ ad reale quid conducere poſſunt, niſi quatenus impoſſibilitatis ſigna, quæ involvunt, ſeſe mutuo deſtruunt.

§. 79. Sumto $z\sqrt{-1} = \frac{1}{2}\log.\frac{1 + t\sqrt{-1}}{1 - t\sqrt{-1}} = \log.\sqrt{\frac{1 + t\sqrt{-1}}{1 - t\sqrt{-1}}}$:

quoniam eſt $\log.\sqrt{\frac{1+v}{1-v}} = v + \frac{1}{3}v^3 + \frac{1}{5}v^5 + \frac{1}{7}v^7 + \frac{1}{9}v^9 + \ldots$ (§. 61.)

erit $z\sqrt{-1} = t\sqrt{-1}\,(1 - \frac{1}{3}tt + \frac{1}{5}t^4 - \frac{1}{7}t^6 + \frac{1}{9}t^8 - \ldots)$

et $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \ldots$

quæ deductio exempli loco ſit utilitatis harum expreſſionum imaginariarum.

Ad hanc expreſſionem etiam conducit exponens differentialis $\frac{dz}{dt} = \frac{1}{1+tt}$ (§. 16.)

Inde enim eſt $\frac{dz}{dt} = 1 - tt + t^4 - t^6 + t^8 - t^{10} + \ldots$

unde $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \frac{1}{11}t^{11} + \ldots$

Porro hæc expreſſio immediate etiam ex primis limitum principiis deducitur modo ſequenti:

Quoniam (§. ac. *Introd.*) $\text{ſin.}\varphi = \frac{(\text{coſ.}n\varphi + \text{ſin.}n\varphi\sqrt{-1})^{\frac{1}{n}} - (\text{coſ.}n\varphi - \text{ſin.}n\varphi\sqrt{-1})^{\frac{1}{n}}}{2\sqrt{-1}}$

$$n\,\text{ſin.}\varphi = n\frac{(\text{coſ.}n\varphi + \text{ſin.}n\varphi\sqrt{-1})^{\frac{1}{n}} - (\text{coſ.}n\varphi - \text{ſin.}n\varphi\sqrt{-1})^{\frac{1}{n}}}{2\sqrt{-1}}$$

$$= n\Big(\frac{1}{n}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-1}\,\text{ſin.}\,n\varphi$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1.2.3}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-3}\,\text{ſin.}^3 n\varphi$$

$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-4\right)}{1.2\ldots4}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-5}\,\text{ſin.}^5 n\varphi$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-6\right)}{1.2\ldots6}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-7}\,\text{ſin.}^7 n\varphi$$

$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-8\right)}{1.2\ldots8}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-9}\,\text{ſin.}^9 n\varphi$$

$$- \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\ldots\left(\frac{1}{n}-10\right)}{1.2\ldots10}\,\text{coſ.}\,n\varphi^{\frac{1}{n}-11}\,\text{ſin.}^{11} n\varphi$$

$+$ - - - -

$-$ - - - -

= coſ.

$$= \cos. n\varphi^{\frac{1}{n}} \Big(\tang. n\varphi - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{1.2.3} \tang.^3 n\varphi + \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(4-\frac{1}{n}\right)}{1.2\ldots5} \tang.^5 n\varphi - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(6-\frac{1}{n}\right)}{1.2\ldots7} \tang.^7 n\varphi + \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(8-\frac{1}{n}\right)}{1.2\ldots9} \tang.^9 n\varphi - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(10-\frac{1}{n}\right)}{1.2\ldots11} \tang.^{11} n\varphi \pm \text{- - - -} \Big)$$

unde facto $n\varphi = z$, ſeu $\varphi = \frac{1}{n} z$ eſt

$$n \sin. \frac{1}{n} z = \cos. z^{\frac{1}{n}} \Big(\tang. z - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{1.2.3} \tang.^3 z + \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(4-\frac{1}{n}\right)}{1.2\ldots5} \tang.^5 z - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(6-\frac{1}{n}\right)}{1.2\ldots7} \tang.^7 z + \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(8-\frac{1}{n}\right)}{1.2\ldots9} \tang.^9 z - \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\ldots\left(10-\frac{1}{n}\right)}{1.2\ldots11} \tang.^{11} z \pm \text{- - - -} \Big)$$

Quo-

Quoniam æquatio hæc ſemper locum habet, etiam limites membrorum ejus ſunt inter ſe æquales; ſed creſcente n hi limites ſunt reſpective

$$z \;\&\; \text{tang.}\,z - \tfrac{1}{3}\text{tang.}^3 z + \tfrac{1}{5}\text{tang.}^5 z - \tfrac{1}{7}\text{tang.}^7 z + \tfrac{1}{9}\text{tang.}^9 z - \ldots$$

ergo $z = \text{tang.}\,z - \tfrac{1}{3}\text{tang.}^3 z + \tfrac{1}{5}\text{tang.}^5 z - \tfrac{1}{7}\text{tang.}^7 z + \tfrac{1}{9}\text{tang.}^9 z - \ldots$

Exempla. Sit p quadrans circumferentiæ, & $z = \tfrac{1}{2}p = 45^\circ$: erit tang. $z = 1$; unde $\tfrac{1}{2}p = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \ldots$

Sit $z = \tfrac{1}{3}p = 30^\circ$: $t = \tfrac{1}{\sqrt{3}}$;

$$\tfrac{1}{3}p = \tfrac{1}{\sqrt{3}}\left(1 - \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{1}{5}\cdot\tfrac{1}{3^2} - \tfrac{1}{7}\cdot\tfrac{1}{3^3} + \tfrac{1}{9}\cdot\tfrac{1}{3^4} - \tfrac{1}{11}\cdot\tfrac{1}{3^5} + \ldots\right.$$

Obſervatio. Poſterior ſeries, quæ priore longe promtius convergit, inprimis ſuit computationi circumferentiæ circuli applicata. Ea uſus Cl. MACHINUS peripheriam circuli diametri = 1 ad centum usque figuras decimales computavit; & poſt illum celeb. LAGNY ad 127 figuras usque eundem calculum produxit (adjutus tamen aliis etiam compendiis, quæ non ſatis explicuit. Vid. *Memoires de l'Academie des Sciences de Paris*, 1719.). Bibliothecam Oxonienſem Bodleianam manuſcriptum poſſidere ex epiſtola D. HORNSBY refert ill. KÆSTNER (*Anfangsgründe der Arithm. und Geometrie*, *5te Aufl.* 1792.): quod ad 156 usque figuras decimales continuatum rationis peripheriæ ad diametrum exponentem exhibeat; & in 113tia cyphra D. LAGNY 8 loco 7 ſubſtitui debere moneat.

§. 80. Series $z = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \tfrac{1}{7}t^7 + \tfrac{1}{9}t^9 - \ldots$ eo citius convergit, ideoque uſui eo magis fit accommodata, quo minor eſt t. Cum autem divergat, quando t eſt unitate major: prima ſpecie non videtur omnibus arcubus poſſe applicari; quod tamen locum habet, uti ratiocinio ſequenti patebit.

Et primum quidem ſeries iſta applicatur dimidio circumferentiæ quadranti, ſeu arcui 45°; ubi $t = 1$ (quamvis fatendum ſit, eam tunc lente admodum convergere.)

Quando autem arcus major eſt dimidio quadrante ſeu 45°: tangens ejus major eſt unitate; ſed tangens complementi ipſius eſt unitate minor, ac proinde complementum prioris arcus computari poteſt per ſeriem convergentem.

Clariſs. Eulerus artificium tradit, quo prædicta ſeries fit uſui quam maxime accommodata; & quod e re eſſe cenſeo hic breviter exponere.

Lemma. Eſto arcus, cujus tangens ſit pars quæpiam aliquota radii. Arcus hic dividi poteſt in duos alios arcus, quorum tangentes pariter erunt radii partes aliquotæ, eæque minores.

Sit $\frac{1}{T}$ tangens alicujus arcus, exiſtente T numero integro poſitivo. Arcus hic dividi poteſt in duos alios, quorum tangentes $\frac{1}{t}$, $\frac{1}{t'}$ etiam erunt partes aliquotæ radii, eæque minores.

Etenim quoniam $\frac{1}{T}$ eſt tangens arcus, qui eſt ſumma arcuum, quorum tangentes ſunt $\frac{1}{t}$ & $\frac{1}{t'}$; erit $\frac{1}{T}=\frac{\frac{1}{t}+\frac{1}{t'}}{1-\frac{1}{tt'}}$: unde (facili calculo) deducitur $(t-T)(t'-T) = TT+1$; hinc $t-T=\frac{TT+1}{t'-T}$. Ut t ſit numerus integer, fiat denominator $t'-T$ æqualis diviſoribus numeratoris $TT+1$; erit factum.

Exempla. Sit $T=1$, ſeu arcus propoſitus ſit dimidius quadrans. $TT+1=2$; $\begin{matrix} t'-T=1,2 \\ t-T=2,1 \end{matrix}$; $\begin{matrix} t'=2,3 \\ t=2,2 \end{matrix}$. Dimidius igitur quadrans dividitur in duos arcus, quorum tangentes ſunt $\frac{1}{2}$ & $\frac{1}{3}$ radii. Summa horum arcuum, ſerie præcedenti promtius convergente computatorum, erit dimidius quadrans.

Sit nunc $T=2$; $TT+1=5$; $\begin{matrix} t'-T=1,5 \\ t-T=5,1 \end{matrix}$; $\begin{matrix} t'=3,7 \\ t=7,3 \end{matrix}$. Quadrans ideo dimidiæ circumferentiæ dividitur in tres arcus, quorum duorum mutuo æqualium tangens eſt $\frac{1}{3}$, & tertii tangens eſt $\frac{1}{7}$.

Sit tandem $T=3$; $TT+1=10$; $\begin{matrix} t'-T=1,2,5,10 \\ t-T=10,5,2,1 \end{matrix}$; $\begin{matrix} t'=4,5,8,13 \\ t=13,8,5,4 \end{matrix}$. Proinde quadrans dimidiæ circumferentiæ dividitur v. gr. in quinque arcus, ut ſequitur: $\frac{1}{2}p = 2$ arc. tang. $\frac{1}{5} + 1$ arc. tang. $\frac{1}{7} + 2$ arc. tang. $\frac{1}{8}$.

Sufficiat, hujus methodi principium breviter expoſuiſſe. Qui plura voluerit, adeat *Commentarios Acad. imp. Petrop. ad ann.* 1737. & Bertrand *Developement de la partie elementaire des Mathematiques.*

§. 81. Formulæ differentiales capite hoc traditæ (§. 76.) felici ſucceſſu ad inveſtigandas proprietates linearum transcendentium, functiones circulares involventium, adhibentur. Sufficiat, exempli loco de cycloïde dicere, cujus tam frequentes & eximiæ in matheſi mixta applicationes occurrunt.

Sit AB axis, & BD baſis cycloïdis AMD. Sit ANB ſemi-circulus ſuper axe AB tanquam diametro deſcriptus; & recta MP axi ordinatim applicata circumferentiæ hujus in N occurrat. Eſt $MN =$ arc. AN. Sit $AB = 2r$, $AP = x$, $MP = y$; erit itaque $y = \sqrt{(2rx - xx)} + r\,\text{arc. ſin. v.} \frac{x}{r}$: unde Fig. 18.a

$\frac{dy}{dx} = \frac{r - x}{\sqrt{(2rx - xx)}} + \frac{r}{\sqrt{(2rx - xx)}} = \frac{2r - x}{\sqrt{(2rx - xx)}} = \sqrt{\frac{2r - x}{x}} = \frac{\sqrt{(2rx - xx)}}{x} =$ $\frac{NP}{AP} = \text{cot.} ANP$. Sit MT tangens cycloidis in M; eſt $\frac{dy}{dx} = \text{cot.} TMP$ (§. 41): quare cot. $TMP =$ cot. ANP; $TMP = ANP$; & proinde MT eſt ipſi NA parallela.

Si eſſet $y = \sqrt{(2rx - xx)} + R\,\text{arc. ſin. v.} \frac{x}{r}$; eſſet
$\frac{dy}{dx} = \frac{r - x}{\sqrt{(2rx - xx)}} + \frac{R}{\sqrt{(2rx - xx)}} = \frac{R + r - x}{\sqrt{(2rx - xx)}}$: unde cycloïdum contractarum & protractarum tangentes determinantur.

Caput Octavum.

De ſumma et differentia duarum quantitatum exponentialium in factores reſolvenda.

§. 82.

Summæ & differentiæ $e^x \pm e^y$ duarum quantitatum exponentialium e^x, e^y, in factores reſolutio tanti eſt in calculis ſuperioribus momenti, & adeo fœcundas ſuppeditat applicationes, ut ei evolvendæ merito mathematici ſtuduerint; quos inter eminet celeb. Eulerus in *Introductione ad Analyſin infinitorum* aliasque. Cum autem Euleriana methodus notione infiniti innixa minus mihi arrideret; optabam, ut reſolutio hæc ad genuina & elementaria limitum principia poſſet reduci. Quod me præſtitiſſe opinor in diſſertatione inſcripta: *Sur la Decompoſition*

tion de la ſomme & de la difference de deux puiſſances à expoſans quelconques de la baſe des logarithmes hyperboliques, dans le but de degager cette decompoſition de toute idée de l'infini. (*Memoires de l'Acad. de Berlin*, 1787—1788.) Et cum diſſertationis hujus objectum cum calculorum ſuperiorum principiis ita arcte cohæreat, ſtrictim illud hic exponere e re eſſe cenſui.

§. 83. Per theorema Coteſianum (§. *ad Introd.*) eſt

$$\begin{aligned}
\left(1+\frac{x}{2n}\right)^{2n}+\left(1-\frac{x}{2n}\right)^{2n} = {} & \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{1}{2n}\pi+\left(1-\frac{x}{2n}\right)^2 \\
& \times \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{3}{2n}\pi+\left(1-\frac{x}{2n}\right)^2 \\
& \times \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{5}{2n}\pi+\left(1-\frac{x}{2n}\right)^2 \\
& \vdots \\
& \times \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{2n-1}{2n}\pi+\left(1-\frac{x}{2n}\right)^2. \\
= {} & 2\left(\left(1-\text{coſ.}\frac{1}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\text{coſ.}\frac{1}{2n}\pi\right)\right) \\
& \times 2\left(\left(1-\text{coſ.}\frac{3}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\text{coſ.}\frac{3}{2n}\pi\right)\right) \\
& \times 2\left(\left(1-\text{coſ.}\frac{5}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\text{coſ.}\frac{5}{2n}\pi\right)\right) \\
& \vdots \\
& \times \left(\left(1-\text{coſ.}\frac{2n-1}{2n}\pi\right)+\frac{xx}{4nn}\left(1+\text{coſ.}\frac{2n-1}{2n}\pi\right)\right)
\end{aligned}$$

$$\begin{aligned}
= 4^n\,\text{ſin.}^2\frac{1}{2n}p\,.\,\text{ſin.}^2.\frac{3}{2n}p\,.\,\text{ſin.}^2.\frac{5}{2n}p\ldots\text{ſin.}^2\frac{2n-1}{2n}p \times {} & \left(\left(1+\frac{xx}{4nn}\text{cot.}^2\frac{1}{2n}p\right)\right. \\
& \times \left(1+\frac{xx}{4nn}\text{cot.}^2\frac{3}{2n}p\right) \\
& \times \left(1+\frac{xx}{4nn}\text{cot.}^2\frac{5}{2n}p\right) \\
& \vdots \\
& \left.\times \left(1+\frac{xx}{4nn}\text{cot.}^2\frac{2n-1}{2n}p\right)\right)
\end{aligned}$$

Hinc

Hinc (§. ae. *Introd.*) $\dfrac{\left(1+\frac{x}{2n}\right)^{2n}+\left(1-\frac{x}{2n}\right)^{2n}}{2}=\Big(\left(1+\frac{xx}{4nn}\cot.^2\frac{1}{2n}p\right)$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{3}{2n}p\right)$$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{5}{2n}p\right)$$

$$\vdots$$

$$\times\left(1+\frac{xx}{4nn}\cot.^2\frac{2n-1}{2n}p\right)\Big)$$

Quoniam autem hæc æquatio ſemper obtinet, limites etiam membrorum ejus ſunt inter ſe æquales. Atqui aucto n (§. 57.) limes prioris membri eſt $\frac{e^x+e^{-x}}{2}$; & (§. 75.) limites factorum $1+\frac{xx}{4nn}\cot.^2\frac{1}{2n}p$ ſunt reſpective $1+\frac{xx}{pp}$

$$1+\frac{xx}{4nn}\cot.^2\frac{3}{2n}p \qquad 1+\frac{xx}{9pp}$$

$$1+\frac{xx}{4nn}\cot.^2\frac{5}{2n}p \qquad 1+\frac{xx}{25pp}$$

$$1+\frac{xx}{4nn}\cot.^2\frac{7}{2n}p \qquad 1+\frac{xx}{49pp}$$

$$\vdots \qquad \vdots$$

Ergo (§. 21.) limes poſterioris membri eſt

$$\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{25pp}\right)\left(1+\frac{xx}{49pp}\right)\ldots$$

Proinde $\frac{e^x+e^{-x}}{2}=\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{25pp}\right)\left(1+\frac{xx}{49pp}\right)\ldots$

§. 84. Per theorema Coteſianum (§. *ad. Introd.*) eſt

$$\left(1+\frac{x}{2n}\right)^{2n}-\left(1-\frac{x}{2n}\right)^{2n}=\Big(\left(1+\frac{x}{2n}\right)^2-\left(1-\frac{x}{2n}\right)^2\Big)\Big(\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{2}{2n}\pi+\left(1-\frac{x}{2n}\right)^2\Big)$$

$$\times\left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{4}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

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$$\times\ \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{6}{2n}\pi+\left(1-\frac{x}{2n}\right)^2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\times\ \left(1+\frac{x}{2n}\right)^2-2\left(1+\frac{x}{2n}\right)\left(1-\frac{x}{2n}\right)\text{coſ.}\frac{2n-2}{2n}\pi+\left(1-\frac{x}{2n}\right)^2\Big).$$

$$=\frac{2x}{n}\times 2\Big(\Big(1-\text{coſ.}\frac{2}{2n}\pi+\frac{xx}{4nn}\Big(1+\text{coſ.}\frac{2}{2n}\pi\Big)\Big)$$

$$\times\ 2\Big(1-\text{coſ.}\frac{4}{2n}\pi+\frac{xx}{4nn}\Big(1+\text{coſ.}\frac{4}{2n}\pi\Big)\Big)$$

$$\times\ 2\Big(1-\text{coſ.}\frac{6}{2n}\pi+\frac{xx}{4nn}\Big(1+\text{coſ.}\frac{6}{2n}\pi\Big)\Big)$$

$$\vdots \qquad\qquad \vdots$$

$$\times\ 2\Big(1-\text{coſ.}\frac{2n-2}{2n}\pi+\frac{xx}{4nn}\Big(1+\text{coſ.}\frac{2n-2}{2n}\pi\Big)\Big)$$

$$=\frac{2x}{n}4^{n-1}\text{ſin.}^2\frac{1}{n}p\,\text{ſin.}^2\frac{2}{n}p\,\text{ſin.}^2\frac{3}{n}p\ldots\text{ſin.}^2\frac{n-1}{n}p\Big(\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{1}{n}p\Big)$$

$$\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{2}{n}p\Big)$$

$$\times\ \Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{3}{n}p\Big)$$

$$\vdots \qquad\qquad \vdots$$

$$\times\ \Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{n-1}{n}p\Big)\Big)$$

$$=(\text{§. ae. Introd.})\ 2x\Big(\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{1}{n}p\Big)$$

$$\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{2}{n}p\Big)$$

$$\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{3}{n}p\Big)$$

$$\vdots \qquad\qquad \vdots$$

$$\Big(1+\frac{xx}{4nn}\text{cot.}^2\frac{n-1}{n}p\Big)\Big).$$

Cum hæc æquatio ſemper obtineat, limites etiam membrorum ejus ſunt inter ſe æquales (§. 4.). Atqui creſcente n limes differentiæ

(1 +

$\frac{\left(1+\frac{x}{2n}\right)^{2n}-\left(1-\frac{x}{2n}\right)^{2n}}{2}$ eſt $\frac{e^x - e^{-x}}{2}$ (§. 57.) Et limites factorum

$\left(1+\frac{xx}{4nn}\cot.^2\frac{1}{n}p\right)$ ſunt reſpective (§. 75.) $1+\frac{xx}{pp}$

$$1+\frac{xx}{4nn}\cot.^2\frac{2}{n}p \qquad\qquad 1+\frac{xx}{4pp}$$

$$1+\frac{xx}{4nn}\cot.^2\frac{3}{n}p \qquad\qquad 1+\frac{xx}{9pp}$$

$$1+\frac{xx}{4nn}\cot.^2\frac{4}{n}p \qquad\qquad 1+\frac{xx}{16pp}.$$

$$\vdots \qquad\qquad\qquad \vdots$$

Ergo (§. 21.) limes poſterioris membri eſt

$$2x\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{4pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{16pp}\right)\ldots.$$

ergo $\frac{e^x - e^{-x}}{2} = x\left(1+\frac{xx}{pp}\right)\left(1+\frac{xx}{4pp}\right)\left(1+\frac{xx}{9pp}\right)\left(1+\frac{xx}{16pp}\right)\ldots.$

§. 85. Duarum formularum $\frac{e^x \pm e^{-x}}{2}$ in factores reſolutarum ope aliæ etiam formulæ in factores reſolvuntur, quales ſunt $\frac{e^{+x} \pm e^{\pm y}}{2}$.

Etenim 1°. $\frac{e^x + e^y}{2}\left[= e^{\frac{x+y}{2}} \times \frac{e^{\frac{x-y}{2}} + e^{-\frac{x-y}{2}}}{2}\right] =$

$$= e^{\frac{x+y}{2}}\left(1+\frac{(x-y)^2}{\pi\pi}\right)\left(1+\frac{(x-y)^2}{9\pi\pi}\right)\left(1+\frac{(x-y)^2}{25\pi\pi}\right.\ldots \text{ (§. 83.)}$$

2°. Hinc $\frac{e^x + e^{-y}}{2} = e^{\frac{x-y}{2}} \times \left(\left(1+\frac{(x+y)^2}{\pi\pi}\right)\left(1+\frac{(x+y)^2}{9\pi\pi}\right)\left(1+\frac{(x+y)^2}{25\pi\pi}\right)\ldots\right.$

3°. $\frac{e^x - e^y}{2}\left[= e^{\frac{x+y}{2}} \times \frac{e^{\frac{x-y}{2}} - e^{-\frac{x-y}{2}}}{2}\right] =$

$$= \frac{x-y}{2}\cdot e^{\frac{x+y}{2}}\left(\left(1+\frac{(x-y)^2}{4\pi\pi}\right)\left(1+\frac{(x-y)^2}{16\pi\pi}\right)\left(1+\frac{(x-y)^2}{36\pi\pi}\right)\ldots\right. \text{ (§. 84.)}$$

4°. Hinc $\frac{e^x - e^{-y}}{2} = \frac{x+y}{2} e^{\frac{x-y}{2}}\left(\left(1+\frac{(x+y)^2}{4\pi\pi}\right)\left(1+\frac{(x+y)^2}{16\pi\pi}\right)\left(1+\frac{(x+y)^2}{36\pi\pi}\right)\ldots\right.$

Hinc

Hinc etiam $\frac{e^{x}+1}{2} = e^{\frac{1}{2}x}\left(\left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{25\pi\pi}\right)\ldots\right.$

$$\frac{e^{x}-1}{2} = \tfrac{1}{2}x\,.\,e^{\frac{1}{2}x}\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\left(1+\frac{xx}{36\pi\pi}\right)\ldots$$

§. 86. Formulæ $e^{x} \pm e^{-x}$ & $e^{y} \pm e^{-y}$ multiplicentur in ſe invicem reſpective.

1°. $(e^{x}+e^{-x})(e^{y}+e^{-y}) = e^{x+y}+e^{x-y}+e^{-(x-y)}+e^{-(x+y)}$; unde facto $\begin{matrix} x+y=v \\ x-y=z \end{matrix}$ formula $e^{v}+e^{-v}+e^{z}+e^{-z}$ in factores reſolvitur.

2°. $(e^{x}+e^{-x})(e^{y}-e^{-y}) = e^{x+y}-e^{-(x+y)}-e^{x-y}+e^{-(x-y)}$; unde formula $e^{v}-e^{-v}-e^{z}+e^{-z}$ etiam in factores reſolvitur.

3°. $(e^{x}-e^{-x})(e^{y}-e^{-y}) = e^{x+y}+e^{-(x+y)}-e^{x-y}-e^{-(x+y)}$; unde formula $e^{v}+e^{-v}-e^{z}-e^{-z}$ etiam in factores reſolvitur.

Aucto factorum numero; hæc reſolutio ad longe plures alias formulas extendi poteſt.

§. 87. Sit etiam formula $e^{x}-2\text{coſ}.2\phi+e^{-x}$ in factores reſolvenda. Juxta §. *af. Introd.* eſt

$$\left(1+\frac{x}{4n+2}\right)^{4n+2}-2\left(1+\frac{x}{4n+2}\right)^{2n+1}\left(1-\frac{x}{4n+2}\right)^{2n+1}\text{coſ}.2\phi+\left(1-\frac{x}{4n+2}\right)^{4n+2}$$

$$=\left(1+\frac{x}{4n+2}\right)^{2}-2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{coſ}.\frac{2\phi}{2n+1}+\left(1-\frac{x}{4n+2}\right)^{2}$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^{2}-2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{coſ}.\frac{2\pi-2\phi}{2n+1}+\left(1-\frac{x}{4n+2}\right)^{2}\right)$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^{2}-2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{coſ}.\frac{2\pi+2\phi}{2n+1}+\left(1-\frac{x}{4n+2}\right)^{2}\right)$$

$$\vdots$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^{2}-2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{coſ}.\frac{2n\pi-2\phi}{2n+1}+\left(1-\frac{x}{4n+2}\right)^{2}\right)$$

$$\times\left(\left(1+\frac{x}{4n+2}\right)^{2}-2\left(1+\frac{x}{4n+2}\right)\left(1-\frac{x}{4n+2}\right)\text{coſ}.\frac{2n\pi+2\phi}{2n+1}+\left(1-\frac{x}{4n+2}\right)^{2}\right)$$

$= 2(1$

$$= 2\Big(1 - \text{coſ.}\,\frac{2\phi}{2n+1} + \frac{xx}{(4n+2)^2}\big(1 + \text{coſ.}\frac{2\phi}{4n+1}\big)$$

$$\times 2\Big(1 - \text{coſ.}\frac{2\pi - 2\phi}{2n+1} + \frac{xx}{(4n+2)^2}\big(1 + \text{coſ.}\frac{2\pi - 2\phi}{2n+1}\big)$$

$$\times 2\Big(1 - \text{coſ.}\frac{2\pi + 2\phi}{2n+1} + \frac{xx}{(4n+2)^2}\big(1 + \text{coſ.}\frac{2\pi + 2\phi}{2n+1}\big)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\times 2\Big(1 - \text{coſ.}\frac{2n\pi - 2\phi}{2n+1} + \frac{xx}{(4n+2)^2}\big(1 + \text{coſ.}\frac{2n\pi - 2\phi}{2n+1}\big)$$

$$\times 2\Big(1 - \text{coſ.}\frac{2n\pi + 2\phi}{2n+1} + \frac{xx}{(4n+2)^2}\big(1 + \text{coſ.}\frac{2n\pi + 2\phi}{2n+1}\big)\Big).$$

$$= 4^{2n+1}\text{ſin.}\,\frac{\phi}{2n+1}\,\text{ſin.}^2\frac{\pi-\phi}{2n+1}\,\text{ſin.}^2\frac{\pi+\phi}{2n+1}\ldots\text{ſin.}^2\frac{n\pi-\phi}{2n+1}\cdot\text{ſin.}^2\frac{n\pi+\phi}{2n+2}\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\phi}{2n+1}\Big)$$

$$\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\pi-\phi}{2n+1}\Big)$$

$$\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\pi+\phi}{2n+1}\Big)$$

$$\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{2\pi-\phi}{2n+1}\Big)$$

$$\vdots \qquad\qquad \vdots$$

$$\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{n\pi-\phi}{2n+1}\Big)$$

$$\times\Big(1+\frac{xx}{(4n+2)^2}\text{cot.}^2\frac{n\pi+\phi}{2n+1}\Big)$$

$$= (\text{§. } ag. \textit{Introd.})\; 4\,\text{ſin.}^2\phi \times \Big(1 + \frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\phi}{2n+1}\Big)$$

$$\times\Big(1 + \frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\pi-\phi}{2n+2}\Big)$$

$$\times\Big(1 + \frac{xx}{(4n+2)^2}\text{cot.}^2\frac{\pi+\phi}{2n+1}\Big)$$

$$\vdots \qquad\qquad \vdots$$

$$\times\Big(1 + \frac{xx}{(4n+2)^2}\text{cot.}^2\frac{n\pi-\phi}{2n+1}\Big)$$

$$\times\Big(1 + \frac{xx}{(4n+2)^2}\text{cot.}^2\frac{n\pi+\phi}{2n+1}\Big).$$

Cum

Cum hæc æquatio ſemper obtineat, etiam limites membrorum ejus ſunt inter ſe æquales (§. 4.). Unde (§§. 57. 75. 21.)

$$e^x - 2\cos.2\varphi + e^{-x} = 4\sin.\varphi \times \left(\left(1 + \frac{xx}{4\varphi\varphi}\right)\right.$$
$$\times \left(1 + \frac{xx}{4(\pi-\varphi)^2}\right)$$
$$\times \left(1 + \frac{xx}{4(\pi+\varphi)^2}\right)$$
$$\times \left(1 + \frac{xx}{4(2\pi-\varphi)^2}\right)$$
$$\times \; - \; - \; -$$
$$\times \; - \; - \; -$$

Eadem expreſſio eodem modo deducitur ex reſolutione in factores formulæ $a^{4n} - 2a^{2n}b^{2n}\cos.2\varphi + b^{2n}$.

Scholium. Quoniam applicationes formularum hoc capite demonſtratarum (tam ad ſummationes plurimarum ſerierum, quam ad varias expreſſiones functionum circularium & logarithmicarum) ab Eulero aliisque mathematicis ita traditæ ſunt, ut nihil, quod deſiderari poſſit, relinquant: mihi ſufficiat, eas ad vera ſua principia revocaſſe, & deductionem ipſarum ab indeterminata & obſcura infiniti notione liberaſſe. Vid. inter alios Euleri *Introductio.* Cap. IX. X. XI.

Caput Nonum.

De infinito, quod vocant, mathematico.

§. 88.

Fig. 11. Sit elipſis conica, cujus æquatio eſt $yy = \frac{bb}{aa}(aa - xx)$, abſciſſis axis ex centro ſumtis. Ducta tangente MT, quæ axi in T occurrat, fit $PT = -(\frac{aa}{x} - x)$ (§. 42.). Fiat $x = 0 = 0 \times a$; erit $PT = -(\frac{aa}{a \times 0} - 0) = -\frac{a}{0} = -\frac{1}{0} \times a$.

Gene-

Generatim. Sit ellipſis cujusvis ordinis, cujus æquatio $y^m = \frac{b^n}{a^m}(a^m - x^m)$: erit $PT = -(a \times \frac{a^{m-1}}{x^{m-1}})$; & facto $x = 0 \times a$, erit $PT = -a \times \frac{1}{0^{m-1}}$.

Cum expreſſiones $\frac{1}{0}$, $\frac{1}{0^m}$ frequentiſſime a mathematicis uſurpentur, & in calculis præſertim ſuperioribus ſæpius occurrant: e re eſſe cenſeo (priusquam ulterius progrediar) veram horum ſymbolorum ſignificationem expendere, & meam de illis ſententiam profiteri; eo magis quod de indole ipſorum inter mathematicos non omnino convenit, pluresque etiam inſignes & aliunde ſummopere commendandi ſcriptores opinionem de illis amplexi ſunt, quæ cum ſenſu communi pugnare videtur. Quod ut faciam, quantum potero, accurate & luculenter, a familiari exemplo ſequente ordiar.

§. 89. Duo viatores *A* & *B*, dato intervallo *D* a ſe invicem diſtantes, in eadem recta, datis velocitatibus, verſus eandem plagam progrediantur; ita ut unus eorum *A* alterum *B* perſequatur. Tres heic, quod ad rationem velocitatum viatorum attinet, diſtinguendi ſunt caſus.

1°. Viator *A* celerius progrediatur quam viator *B*. Hoc caſu intervallum, quo duo viatores a ſe invicem primum diſtabant, continue minuitur, donec *A* ipſum *B* attingat.

2°. Ambo viatores æqualibus progrediantur velocitatibus. Hoc caſu intervallum, quo duo viatores primum diſtabant, idem ſemper manet; utcunque diu progrediantur, & utcunque magnam viam ambo emetiantur.

3°. Viator *A* lentius progrediatur altero *B*. Hoc caſu intervallum, quo duo viatores primum diſtabant, continue increſcit; & tanto majus fit, quo diutius incedunt. Iidem viatores aut potuerunt anterius proficiſci ex uno eodemque puncto, ad plagam ſito oppoſitam ei, juxta quam progrediuntur; aut illic ſibi jam mutuo occurriſſe; aut denique in eo puncto ſibi mutuo occurrerent, ſi uterque retrogrederetur.

Hinc dato intervallo, quo duo viatores a ſe invicem diſtant, datisque velocitatibus, quibus progrediuntur; ſi quæratur locus occurſus ipſorum mutui: prius quam quæſtionis propoſitæ inveſtigatio ſuſcipiatur, inquirendum eſt, utrum poſſi-

poſſibilis ſit, nec ne; ſeu inquirendum eſt in rationem velocitatum, qua prædicta poſſibilitas aut impoſſibilitas determinatur. Algebriſta autem non ſemper ita caute procedit, neque ſemper prævium hoc examen inſtituit. Univerſalitatis (nimio quandoque) ſtudio impulſus formulas quærit generales, quas ad omnes omnino caſus flectere ſatagit, & iis etiam applicare, ad quos non quadrant; quod hoc ipſo exemplo patebit.

Sint nimirum V, v, velocitates datæ duorum viatorum A, B, intervallo $= D$ primitus invicem diſtantium, & juxta eandem plagam progredientium, ita ut A ipſum B perſequatur. Velocitate V poſita majore quam v; quando viator A alterum B attingit, erit

via ab B percurſa $D \times \frac{v}{V-v}$

- - A - - $D \times \frac{V}{V-v}$; & formulæ hæ apte omnino repræſentant caſum, quo $V > v$.

Algebriſta autem, omnes univerſim caſus una eademque formula complecti ſatagens, quærit etiam: quid juxta eas fiat, ſi $V \leqq v$?

Caſu autem, quo $V = v$, expreſſiones ſpatiorum ab utroque viatore percurſorum transformantur in hanc $D \times \frac{1}{0}$; cujus ſignificationem definire oportet.

Ut cum ſenſu communi reſponſum hoc algebræ conveniat, aliter illud interpretari non poſſum, niſi dicendo: illud indicare excluſionem ſeu impoſſibilitatem occurſus mutui viatorum. Algebraiſta autem de occurſu mutuo loqui pergens, dum de illo agi amplius non poteſt, viatores ſibi mutuo occurrere in *diſtantia infinita* pronuntiat. Quæ loquendi forma rite explicata nihil aliud ſignificare poteſt, quam locum occurſus mutui viatorum hoc caſu aſſignari non poſſe; nec ullibi unquam viatores ſibi mutuo occurrere. Symbolum igitur $\frac{1}{0}$, quod formulis præcedentibus caſui huic applicatis introducitur, de impoſſibilitate quæſtionis nunc propoſitæ nos monet, eodem prorſus modo, quo ſigna $\sqrt{-1}$, $\sqrt[2n]{-1}$ impoſſibilitatem docent ſolutionis & repugnantiam mutuam conditionum problematum, quorum tanquam poſſibilium tractatio algebraica ſigna illa introducere coëgit.

Quam-

Quamdiu $V > v$; diſtantia occurſus viatorum a locis, e quibus egredi ponuntur, pendet ab intervallo D horum locorum. Facto autem $V = v$, impoſſibilitas occurſus mutui eadem manet, quodcunque ſit intervallum D, quo primitus ſejungebantur; & abſurdum eſſet dicere: impoſſibilitatem hanc fieri duplam aut triplam, prouti intervallum illud ſit duplo aut triplo majus.

Quod ad tertium caſum attinet, quo $V < v$; formulæ præcedentes fiunt

$$\begin{matrix} D \times \dfrac{v}{-(v-V)} \\ D \times \dfrac{V}{-(v-V)} \end{matrix}, \text{ quæ transformantur in has } \begin{matrix} -D \times \dfrac{v}{v-V} \\ -D \times \dfrac{V}{v-V} \end{matrix},$$

& indicant: punctum occurſus mutui nunc jacere ad plagam oppoſitam ei, verſus quam viatores progrediuntur; neque in parte anteriori, ſed in parte poſteriori eſſe quærendum. Quoniam autem caſus hic ad præcipuum tractationis præſentis objectum non pertinet, breviter de eo in ſequentibus agetur.

Exemplum allatum adeo luculentum eſt, ut ſupervacaneum foret diutius rei huic immorari, niſi ea, utut per ſe ſimplex, pluribus inanium loquendi modorum tricis involuta fuiſſet. Liceat itaque idem exemplum ſub forma tantillum diverſa, eaque geometrica, exponere.

Problema generale hoc eſt: datam rectam in ratione data producere.

Sit AB recta magnitudine data, producenda ad X usque, in directione Fig. 19. 1°.
quæ tendit ab A verſus B; ita ut rectæ AX, BX ſint inter ſe in ratione data, quam habent lineæ a & b.

Conſtructio. Ex punctis A, B ad easdem rectæ AB partes ducantur duæ rectæ Aa, Bb, ſibi invicem parallelæ, & in data ratione linearum a, b. Jungatur ab recta, quæ (ſi fieri poſſit) datæ AB productæ occurrat in X. Hoc erit punctum quæſitum.

Ut punctum X jaceat verſus plagam propoſitam, oportet, ſit $Aa > Bb$. Sed ſi $Aa < Bb$, recta ba occurrit datæ AB verſus plagam priori oppoſitam productæ.

Sit autem $Aa = Bb$. Quadrilateri $AabB$ lateribus Aa, Bb poſitis ſibi invicem Fig. 19. 2°.
æqualibus & parallelis, quadrilaterum hoc eſt parallelogrammum (El. I. 33.); proinde lineæ AB, ab ſibi mutuo occurrere non poſſunt. Quod ſi parallelæ eſſe

dicantur rectæ, quæ ſibi mutuo in diſtantia infinita occurrunt, idem erit, ac ſi dicatur: eas ſibi mutuo nullibi, ſeu non occurrere.

Fig. 19. 1°. Ut calculus conſtructioni præcedenti applicetur, datæ AB per punctum b parallela agatur, quæ rectæ Aa in a' occurrat. Triangula $aa'b$, aAX, bBX ſunt invicem ſimilia; proinde $\begin{matrix} aa' : aA = ba' : AX \\ aa' : bB = ba' : BX \end{matrix}$; ſeu $\begin{matrix} a-b : a = AB : AX \\ a-b : b = AB : BX \end{matrix}$;

unde $\begin{matrix} AX = AB \times \dfrac{a}{a-b} \\ BX = AB \times \dfrac{b}{a-b} \end{matrix}$.

Cum calculus hic nitatur triangulorum $aa'b$, aAX, bBX ſimilitudine, omnino exiſtentiam eorum triangulorum ſupponit; quæ cum non exiſtant, quando punctum a' incidit in punctum a, ſeu quando $Aa = Bb$; mirum videri non debet, coactam calculi ad hunc caſum applicationem, utpote ſundamento ſuo deſtitutam, ad impoſſibile deducere.

Si eſſet punctum occurſus eo caſu, quo Aa & Bb inter ſe ſunt æquales; eſſent etiam AX & BX invicem æquales: cum tamen quantitate data AB differre ſupponantur. Abſurdum hoc non moratur infiniti fautores, qui duas magnitudines infinitas invicem æquales manere ſtatuunt, ſi una illarum (vel utraque) finitam patiatur mutationem. Genuinus formulæ hujus loquendi ſenſus alius eſſe non poteſt, quam hic: *Sint duæ quantitates mutabiles, quarum utraque major reddi poteſt quacunque quantitate propoſita, & data ſit differentia prædictarum quantitatum; ratio æqualitatis limes eſt rationis decreſcentis majoris ad minorem.* Hoc patet ex ipſa conſtructione præcedente. Data enim ratione quacunque $Aa : Bb$ majoris ad minorem, utut parum ab ratione æqualitatis differat; fiat $Bb' \begin{matrix} > Bb \\ < Aa \end{matrix}$, ſeu fiat $Aa : Bb' \begin{matrix} > 1 : 1 \\ < Aa : Bb \end{matrix}$; ducta ab' occurret ipſi AB productæ in X'; & fiet $AX' : BX' = Aa : Bb' \begin{matrix} > 1 : 1 \\ < Aa : Bb \end{matrix}$. Idemque facile methodo mere algebraica demonſtratur.

Fig. 19. 2°. Negatio occurſus linearum AB, ab nititur æqualitate linearum Aa, Bb, ſeu valore (180°) ſummæ angulorum BAa, baA; nec ad ejus excluſionem quicquam confert magnitudo lineæ AB: & quemadmodum idea parallelismi duarum

rum rectarum unica eſt, ita etiam dici nequit impoſſibilitatem occurſus mutui duarum linearum *AB*, *ab* majorem eſſe aut minorem, prouti linea *AB* eſt major aut minor; ſeu ſignum impoſſibilitatis $\frac{1}{0}$ eſt ſignum unicum, quæcunque ſit quantitas *a*, quacum per multiplicationis ſignum conjungatur; eodem prorſus modo, quo omnia ſigna imaginaria $a\sqrt{-1}$ ad unicum $\sqrt{-1}$ communi mathematicorum conſenſu reducuntur.

§. 90. Geometricam problematis propoſiti (utut elementaris) evolutionem eo lubentius expoſui, quod hæc ipſa quæſtio, aut aliæ ipſi affines, ſub falſo obtutu conſideratæ, variis loquendi modis (meo quidem judicio ſenſu caſſis) anſam præbuerunt. Dicunt v. gr. algebriſtæ, lineam rectam circulum eſſe, cujus radius eſt infinitus. Uno aut altero exemplo indicare ſufficiet, quid ſibi velit ejusmodi loquendi modus.

Sint duo puncta poſitione data. Notum eſt: rectam, quæ bifariam & ad angulos rectos ſecat lineam rectam puncta data jungentem, locum eſſe omnium punctorum ab his punctis æqui-diſtantium. Nempe non modo quodvis punctum in perpendiculo hoc ſitum æqui-diſtat a duobus punctis datis; ſed etiam quodvis punctum extra hoc perpendiculum (in eodem plano) ſitum inæqualiter diſtat ab his punctis.

Notum pariter eſt (vid. Apollonii pergæi *Loca plana* L.II. Prop. 2.): circumferentiam circuli locum eſſe omnium punctorum, quorum diſtantiæ a duobus punctis datis ſunt inter ſe in ratioue data a ratione æqualitatis diverſa.

Duos hosce caſus, utut inter ſe diverſos, & ab antiquis geometris ſedulo ſejunctos, ſub uno complecti ſatagunt algebriſtæ. Cum radius circuli, cujus circumferentia eſt locus propoſitus, eo major fiat, quo ratio data, a ratione æqualitatis diverſa, ad eam propius accedit; & quemadmodum ratio data ad rationem æqualitatis propius accedere poteſt quam ratio quævis propoſita a ratione æqualitatis diverſa, ita etiam radius prædicti circuli quacunque linea data major fieri poſſit: concludunt inde, quod, ſi ratio data fuerit ipſa ratio æqualitatis, radius circuli fiat infinitus. Sed hoc caſu locus propoſitus eſt linea recta; ergo (dicunt,) linea recta eſt circulus, cujus radius infinitus. Cum autem hoc caſu idea circuli neceſſario connectatur cum diſcrepantia rationis datæ a ratione

tione æqualitatis, ſublata hac diſcrepantia tollitur etiam idea circuli. Qui dicit, lineam rectam eſſe circulum, cujus radius eſt infinitus, de circulo loquitur, cujus neque centrum neque radius poſſunt aſſignari; dicit itaque, lineam rectam circulum eſſe non circularem.

Fig. 19. 1°. Scilicet: ſit diſtantia AB punctorum A, $B = D$; & ſit ratio data æqualis rationi $a : b$, a ratione æqualitatis diverſæ. Secetur AB in puncto x, eademque producatur ad X usque in ratione data. Biſecetur Xx in Z; erit Z centrum, & Zx radius circuli propoſiti. Fit $Zx = D \times \frac{ab}{a-b.\,a+b}$

$$AZ = D \times \frac{aa}{a-b.\,a+b}$$

$BZ = D \times \frac{bb}{a-b.\,a+b}$; unde inferunt algebriſtæ, Zx, AZ, BZ infinitas fieri, quando $a=b$; cum dicere debuiſſent, nullum eſſe radium, nullum centrum.

Idem valet de aliis quibusdam locis ad circumferentiam circuli. Sint v. gr. duo puncta poſitione data, ad quæ ex puncto quocunque agantur rectæ; & data ſit differentia ſpatiorum, quæ ad quadrata harum rectarum datas habent rationes, inter ſe inæquales. Locus prædicti puncti etiam eſt circumferentia quædam circuli (Apollonii pergæi *Loca plana* L. II. Prop. 4.). Sed ſi rationes datæ ſint inter ſe æquales, locus puncti illius eſt recta poſitione data. Unde iterum concludunt algebriſtæ, lineam rectam circulum eſſe, cujus radius eſt infinitus; cum antiqui geometræ caſum hunc ab altero diverſum ſeorſim pariter tractaverint.

Atque hic rurſus ſe offert ſimplicitas ideæ impoſſibilitatis. Quemadmodum enim linea recta ita eſt unica, ut omnes lineæ rectæ mutuo congruant; ita etiam impoſſibilitas curvaturæ ſit unica, dum contra nullus eſt limes variorum graduum curvaturæ.

§. 91. Quam de vera ſymboli $\frac{1}{0}$ ſignificatione profeſſus ſum ſententiam, altero exemplo, eoque mere algebraico, confirmabo.

Sit

Sit fractio $\frac{1}{x-a\,.\,x-b} = \frac{1}{a-b}\cdot\frac{1}{x-a}$
$$+\frac{1}{b-a}\cdot\frac{1}{x-b}.$$

Hinc per fractionem $\frac{1}{x-a}$ utrinque multiplicando, fit

$$\begin{aligned}
\frac{1}{(x-a)^2\,.\,x-b} &= \frac{1}{a-b}\cdot\frac{1}{x-a^2} \\
&\quad + \frac{1}{b-a}\cdot\frac{1}{x-a\,.\,x-b} \\
&= \frac{1}{a-b}\cdot\frac{1}{x-a^2} \\
&\quad - \frac{1}{a-b^2}\cdot\frac{1}{x-a} \\
&\quad + \frac{1}{a-b^2}\cdot\frac{1}{x-b}.
\end{aligned}$$

Fractio igitur $\frac{1}{x-a^2\,.\,x-b}$, cujus denominator factorem duplicem $x-a^2$ continet, nequit unice reſolvi in fractiones, quarum denominatores factoribus ſimplicibus $x-a$, $x-b$ æquales ſint. Sed reſolutio fractionis illius neceſſario continet fractionem, cujus denominator eſt factor duplex $x-a^2$.

Proponatur autem fractio $\frac{1}{x-a\,.\,x-a'\,.\,x-b}$, cujus factores denominatoris $x-a$, $x-a'$, aut reipſa inæquales ſint, aut tanquam tales tractentur. Erit

$$\begin{aligned}
\frac{1}{x-a\,.\,x-a'\,.\,x-b} &= \frac{1}{a-a'\,.\,a-b}\cdot\frac{1}{x-a} \\
&\quad + \frac{1}{a'-a\,.\,a'-b}\cdot\frac{1}{x-a'} \\
&\quad + \frac{1}{b-a\,.\,b-a'}\cdot\frac{1}{x-b}.
\end{aligned}$$

Tum fiat $a=a'$; erit
$$\begin{aligned}
\frac{1}{x-a\,.\,x-a'\,.\,x-b} &= \frac{1}{a-a'\,.\,a-b}\cdot\frac{1}{x-a} &&= \tfrac{1}{0}\cdot\frac{1}{a\,.\,a-b}\cdot\frac{1}{x-a} \\
&\quad + \frac{1}{a'-a\,.\,a'-b}\cdot\frac{1}{x-a'} &&\quad \pm \tfrac{1}{0}\cdot\frac{1}{a\,.\,a-b}\cdot\frac{1}{x-a} \\
&\quad + \frac{1}{b-a\,.\,b-a'}\cdot\frac{1}{x-b} &&\quad + \frac{1}{b-a^2}\cdot\frac{1}{x-b}
\end{aligned}$$

Pro-

Proinde introductione ſymboli $\frac{1}{0}$ monemur, reſolutionem functionis $\frac{1}{(x-a)^2} \cdot \frac{1}{x-b}$ in fractiones, quarum denominatores ſint factores ſimplices $x-a$, $x-b$, eſſe impoſſibilem, uti jam vidimus.

Hoc caſu, ut ad veram reſolutionem perveniamus, conjungendæ ſunt duæ fractiones $\begin{array}{c} \frac{1}{a-a'.a-b} \cdot \frac{1}{x-a} \\ \frac{1}{a'-a.a'-b} \cdot \frac{1}{x-a'} \end{array}$, conſiderando factores $\frac{a-b.a'-b}{x-a.x-a'}$, tanquam reſpective diverſos. Erit

$$\frac{1}{a-a'} \left\{ \begin{array}{c} \frac{1}{a-b} \cdot \frac{1}{x-a} \\ -\frac{1}{a'-b} \cdot \frac{1}{x-a'} \end{array} \right\} = \frac{1}{a-a'} \times \frac{aa-a'a'-b(a-a')-x(a-a')}{a-b.a'-b.x-a.x-a'} = \frac{a+a'-b-x}{a-b.a'-b.x-a.x-a'}$$

$$= \frac{a-b-(x-a')}{a-b.a'-b.x-a.x-a'} = \begin{array}{c} \frac{1}{a'-b} \cdot \frac{1}{x-a.x-a'} \\ -\frac{1}{a-b.a'-b} \cdot \frac{1}{x-a} \end{array} = \begin{array}{c} \frac{1}{a-b} \cdot \frac{1}{x-a^2} \\ -\frac{1}{a-b^2} \cdot \frac{1}{x-a} \end{array}$$, caſu, quo $a = a'$.

Idem dicatur de aliis fractionum reſolutionibus.

§. 92. Tranſeo ad exemplum geometricum, & rei præſenti omnino accommodatum.

Cum in omni triangulo (rectilineo) angulorum internorum ſumma æqualis ſit duobus rectis: ſi detur ſumma duorum angulorum internorum, ad trianguli poſſibilitatem requiritur, ut ſumma minor ſit duobus rectis. Proinde omnes calculi in triangulis, quorum ſumma duorum angulorum datur, inſtituti cum hac propoſitione fundamentali debent conſentire.

Sit trianguium, cujus latera ſint A, B, C;
& anguli ipſis reſpective oppoſiti a, b, c.

Anguli b & c, & latus A dentur magnitudine; erit

$$B = A \frac{\text{ſin.} b}{\text{ſin.} a} = A \frac{\text{ſin.} b}{\text{ſin.} 180° - (b+c)} = A \frac{\text{ſin.} b}{\text{ſin.} b+c}$$

$C =$ - - $= A \frac{\text{ſin.} c}{\text{ſin.} b+c}$, quæ expreſſiones veræ ſunt, quamdiu $b+c < 180°$.

Sit

Sit $b+c = 180°$; lineæ B & C fiunt parallelæ: ſublato linearum B, C mutuo occurſu, ſimul evaneſcit omnis idea tam anguli a, quam trianguli & laterum illius; & ſimul omnia ruunt fundamenta propoſitionum, quæ nonniſi de triangulis actu exiſtentibus poſſunt prædicari. Algebriſta autem nimio univerſalitatis ſtudio formulas præcedentes etiam nunc conſervare, & ad eum quoque caſum flectere aggreditur, in quem quadrare non poſſunt. Cum ſcilicet poſito $b + c = 180°$, & proinde ſin. $b + c = 0$; formulæ præcedentes fiant

$$B = A\,\frac{\text{ſin.}\,b}{0},\quad C = A\,\frac{\text{ſin.}\,c}{0}$$

pronuntiat: verticem trianguli etiamnum ficti ab latere A infinite diſtare, ipſaque latera B & C infinita fieri. Hic autem loquendi modus (meo quidem judicio) aliter tolerari non poteſt, niſi quatenus impoſſibilitatem afferat trianguli, cujus duorum angulorum ſumma ſit duobus rectis æqualis. Hæc autem impoſſibilitas a valore ſummæ angulorum b, c unice pendet, quæcunque ſit rectæ A ipſis adjacentis magnitudo, & quicunque ſint anguli b, c ſeorſim ſumti. Attamen ſi formulæ, quibus trianguli latera determinantur, huic etiam caſui applicari poſſent, quo deeſt punctum occurſus; impoſſibilitas determinationis laterum trianguli etiamnum ficti diverſa appareret, pro diverſa magnitudine tam lineæ A, quam angulorum b & c, juxta formulas $B = A\frac{\text{ſin.}\,b}{0}$, $C = A\frac{\text{ſin.}\,c}{0}$. Unde ſequeretur, lineas ab uno eodemque puncto iuxta datam directionem ductas eſſe inter ſe magnitudine utlibet diverſas, eo momento, quo infinite eſſe dicuntur.

Idem luculentius etiam patet, quando formulæ, quæ veræ ſunt de ſuperficiebus triangulorum, ſpatiis applicantur, quæ jam non ſunt triangula.

Superficies trianguli, cujus latera & anguli ſunt ut prius $\begin{matrix} A, & B, & C \\ a, & b, & c \end{matrix}$, eſt $\frac{1}{2}A^2 \times \frac{\text{ſin.}\,b\ \text{ſin.}\,c}{\text{ſin.}\,b+c}$; quæ formula caſu, quo $b + c = 180°$, & proinde ſin. b = ſin. c, transformatur in hanc $\frac{1}{2}A^2\,\frac{\text{ſin.}^2\,b}{0}$. Proinde ſi formulas, quæ de ſpatiis finitis veræ ſunt, ſpatiis etiam illimitatis applicare liceret; ſpatia illimitata inter duas

rectas

rectas parallelas extenſa forent in ratione duplicata diſtantiarum harum parallelarum, cum ſpatia hæc inter ſe eſſe in ratione ſimplici harum diſtantiarum tradantur.

§. 93. Ut hactenus dicta, quantum fieri poteſt, diſtincte curvarum doctrinæ poſſint applicari; a curva in elementis conſiderata, circulo nempe, ordiar.

Fig. 20. Sit circulus, cujus centrum C, radius $CA = r$. Ex puncto M ducatur tangens MT, quæ radio CA producto in T occurrat; ſeu ſit MT ipſi CM perpendicularis, atque ex eodem puncto M demittatur MP perpendicularis radio CA. Sit $CP = x$; erit $CT = \frac{rr}{x}$. Expreſſio hæc (ex ſimilitudine triangulorum CMP, CTM deducta) valorem rectæ CT determinat, quamdiu ipſa CT exiſtit. Ut autem proportio $CP : CM = CM : CT$, ſeu $x : r = r : CT$, inſtitui poſſit, neceſſe eſt, ut CP ad CM ſeu x ad r aliquam habeat rationem; ſeu neceſſe eſt, (triangulo CMP exiſtente) linea CP non ſit zero. Poſita igitur $CP = 0 = 0 \times r$, ſymbolum $CT = \frac{rr}{0 \times r} = \frac{r}{0}$ monet non amplius locum dari ſubtangenti. Hoc etiam ex antedictis fluit (§. 92.), cum hoc caſu anguli ACM, CMT ambo ſint recti; ac proinde lineæ CA, MT invicem parallelæ.

Idem valet de ellipſi, cujus æquatio eſt $y = \frac{b}{a}\sqrt{(aa-xx)}$. Itaque ſubtangens $PT = \frac{aa}{x} - x$; unde cot. $PMT = \frac{b}{a} \cdot \frac{x}{\sqrt{aa-xx}}$. Facto $x = 0 = a \times 0$, fit $PT = \frac{a}{0}$, cot. $PMT = \frac{b}{a} \times 0$; unde angulus $PMT = 90°$, & PT determinari nequit.

Ut hæc clarius etiam, ſi fieri poſſit, ob oculos ponantur, regrediamur ad definitionem ſubtangentis, & ad modum, quo determinavimus, eam eſſe limitem (ſi quis ſit) ſubſecantis. Poſuimus (§. 39.) tangentem curvæ diametro, ad quam refertur, occurrere. Intervallum quoddam datum limes eſt datus (juxta definitionem §. 1.), ad quem ſubſecans perpetuo accedit, ſed quem nunquam attingit. Sublato tangentis & diametri occurſu mutuo, fundamenta omnia, quibus rationis, quam ſubſecans habet ad lineam diametro ordinatim applicatam, limitem determinavimus, per ſe labuntur. Linea MQ, diametro & proinde tangenti parallela, tangenti non amplius occurrit. Sublato itaque triangulo

Fig. 11.

gulo MQQ', cujus contemplatione nituntur ratiocinia §. 39.; ipſa etiam hæc ratiocinia & conſequentiæ ex illis deductæ corruunt. Nominatim cadit conſequentia: rectas PT, MT, non amplius determinatas, limites eſſe rectarum PS, MS.

Sit $AP = x$; $PM = y$; $Pp = \Delta x$;

$$mp = y \pm \frac{\Delta x}{1}\cdot\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3} + \ldots.$$

$$\& \quad \Delta y = \pm \frac{\Delta x}{1}\cdot\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3} + \ldots.$$

Itaque $PS\left(=y.\frac{\Delta x}{\Delta y}\right) = y \times \frac{1}{\pm \frac{dy}{dx} + \frac{\Delta x}{1.2}\cdot\frac{ddy}{dx^2} \pm \frac{\Delta x^2}{1.2.3}\cdot\frac{d^3y}{dx^3} + \ldots.}$

Ut aliquis ſit limes rectæ PS, oportet, ſit limes aliquis expreſſionis ipſius; proindeque exponens differentialis $\frac{dy}{dx}$ non ſit zero. Quodſi autem $\frac{dy}{dx} = 0$: ſymbolo $PS = y \times \frac{1}{0}$ monemur, ſubſecantem PS ultra omnem limitem datum creſcere poſſe; de ſubtangente igitur non eſſe loquendum.

Caſus proinde, quo tangens MT & diameter PA invicem ſunt parallelæ, tanquam ſingularis & unicus eſt conſiderandus. Scilicet ex puncto M ducatur recta quæcunque MS, quæ cum tangente MT angulum faciat TMS utcunque exiguum. Recta hæc MS occurrit diametro AP, cum ſumma angulorum SMP, SPM (minor ſumma angulorum TMP, MPA) minor ſit duobus rectis; eademque MS curvæ etiam in altero puncto M' occurrit (cum jam recta MT curvam in puncto M contingere ſupponatur). Imminuto angulo TMS, lineæ MS, PS creſcunt quidem; veruntamen magnitudo illarum definitur per hunc angulum. Neque dici poteſt (juxta definitionem §. 1.) prædictas lineas limitum eſſe capaces, easdemque a lineis magnitudine indeterminatis (ſeu neque datis neque dabilibus) minus differre poſſe, quam ulla quantitate propoſita.

§. 94. Quoniam frequentiſſime a mathematicis uſurpatur formula: parabolam eſſe ellipſin infinitam, ſeu, cujus axis eſt infinitus; e re erit genuinam locutionis hujus ſignificationem declarare: quod pluribus præſtabo modis, ut, quæ de hoc exemplo monuero, facilius poſſint ad alia ſimilia applicari.

Cum nempe modos inter geneſeos ſectionum conicarum tres ſint præcipui, ad quos reliqui facile poſſunt reduci; unumquemque illorum ſeorſim expendam.

1°. Sectio conica referatur ad focum & ad directricem.

Fig. 21. Sit AB recta poſitione data, & ſit F punctum poſitione datum; & deſcribenda ſit ellipſis, cujus focus F & directrix AB, data ratione diſtantiarum ſingulorum ellipſis punctorum a foco F & a directrice AB.

Ex puncto F demittatur in AB perpendicularis FD, quæ ſecetur in S in ratione data; erit S vertex axis transverſi ſectionis, foco F vicinior.

Quoniam ſectio conica propoſita debet eſſe elliptica: ut vertex alter ſectionis ſeu ejusdem axis determinetur, producenda eſt recta DF in S', ita ut $DS' : FS' = DS : SF$; & quoniam DS' debet eſſe major quam FS', oportet, ſit $DS > SF$. Seu ut ſectio propoſita ſit elliptica, ratio data diſtantiarum puncti cujusvis ſectionis a directrice & a foco debet eſſe ratio majoris ad minorem.

Quo poſito erit SS' axis transverſus ellipſis; biſecta autem SS' in C, erit CS dimidius axis transverſus, & CF excentricitas.

Sit $FD = d$; ſitque ratio data $SF : SD = a : b$:

$$\text{erit} \quad SF = d \times \frac{a}{a+b}; \quad SD = \frac{b}{a+b}$$

$$S'F = d \times \frac{a}{a-b}; \quad S'D = d \times \frac{b}{a-b}$$

$$SS' = d \times \frac{2aa}{a-b.a+b}$$

$$CS = d \times \frac{aa}{a-b.a+b}$$

$$CF = d \times \frac{ab}{a-b.a+b}$$

quadratum dimidii axis ſecundi $SF \times S'F = dd \times \dfrac{aa}{a-b.a+b}$.

Quando ratio data $a : b$ eſt ratio æqualitatis: puncta S', C, F' non amplius poſſunt determinari; ſeu punctorum horum poſitiones ſunt impoſſibiles, & rectarum SS', SC, SF' magnitudines pariter fiunt impoſſibiles: ſcilicet curva deſcribenda non amplius in ſe recurrit, ac proinde præcipuum cnrvarum ellipticarum caracterem amittit. Curva itaque hoc caſu deſinit eſſe elliptica; ſed eſt curva

ſui

ſui generis, parabola nempe, quæ neque centrum, neque alterum focum, neque alterum verticem habet. Qui dicit, parabolam eſſe ellipſin, cujus centrum a foco infinite diſtat; dicit, parabolam eſſe ellipſin, cujus centrum aſſignari nequit, ſeu cujus centrum nullibi exiſtit, ſeu quæ centrum non habet. Parabola igitur eſſet ellipſis non elliptica; quod ſenſu caret. Parabola igitur curva eſt non elliptica, & ſui generis.

Impoſſibilitatem poſitionis punctorum S', C, F', & magnitudinis rectarum FS', FC, FF', indicant formulæ præcedentes; quæ, poſito $a = b$, impoſſibilitatis ſigno $\frac{1}{0}$ afficiuntur.

Quemadmodum autem ratio æqualitatis limes eſt rationis decreſcentis majoris ad minorem, ita etiam ratio æqualitatis limes eſt rationis diſtantiarum cujusvis puncti ellipſis a directrice & a foco huic proximo. Unde parabola limes eſt ellipſium eandem directricem eundemque focum poſitione datos habentium, eodem prorſus modo, quo circulus limes eſt figurarum ipſi inſcriptarum aut circumſcriptarum.

2°. Sectionum conicarum originem in ipſo cono contemplabor.

Sit ACA' ſectio coni recti (v. gr.), plano per axem transeunte facta. Sit SMS' coni hujus ſectio plano priori perpendiculari facta, quodque lateribus CA, CA' in punctis S & S' occurrat. Erit igitur ſectio communis utriusque plani SS' axis transverſus ellipſis, & expreſſio axis hujus eſt Fig. 22.

$$CS \times \frac{\text{ſin.}C}{\text{ſin.}S'} = CS \times \frac{\text{ſin.}C}{\text{ſin.}(S'SA - C)} = SS'.$$

Quamdiu angulus $S'SA$ major eſt angulo C: planum SMS' lateri CA' occurrit; proinde tam puncti S' poſitio, quam rectæ SS' magnitudo determinatur, & ſeſtio eſt elliptica; utcunque parum angulus ASS' (major angulo C) ab hoc angulo differat. Atque uti angulus C limes eſt anguli decreſcentis ASS' (quatenus angulus hic major ponitur angulo C); ita etiam parabola, quæ ſectio eſt coni, quando angulus ASS' deſinit eſſe major angulo C, ſeu fit ipſi æqualis, limes eſt ſectionis, quæ reſpondet inæqualitati angulorum ASS' & C: ac ceſſatio ſectionis mutuæ rectarum SS', CA', proindeque impoſſibilitas magnitudinis rectæ SS' denotatur introductione ſigni $\frac{1}{0}$ in expreſſionem lineæ SS', quæ fit

$$SS' = CS \times \frac{\text{ſin.}C}{\text{ſin.}0} = CS \times \frac{\text{ſin.}C}{0}.$$

 Sit

Sit MP recta ipſi SS' perpendicularis, ſeu axi SS' ordinatim applicata; & per MP agatur planum baſi parallelum, quod lateribus coni CA, CA' in punctis B, B', occurrat. Erit $MP^2 = BP \times PB'$. Atqui

$$BP : SP = \text{ſin.}S : \text{ſin.}B$$
$$PB' : S'P = \text{ſin.}S' : \text{ſin.}B'$$

hinc $BP \times PB' : SP \times PS' = \text{ſin.}S \times \text{ſin.}S' : \text{ſin.}B \times \text{ſin.}B'$.

Proportio hæc locum habet, quamdiu punctum S' aſſignatur. Deficiente autem puncto S', & proinde deficiente ſimul triangulo $PB'S'$, proportio $PB' : S'P = \text{ſin.}S' : \text{ſin.}B'$ (quæ trianguli $PS'B'$ exiſtentia nititur) inſtitui non poteſt. Sed caſus hic ſingularis peculiari etiam modo erui debet. Tum ſcilicet recta PB' datur magnitudine; & ex priori proportione $BP : SP = \text{ſin.}S : \text{ſin.}B$ conſequitur $BP \times PB' : SP \times PB' = \text{ſin.}S : \text{ſin.}B = \text{ſin.}C : \text{ſin.}B$
ſeu $MP^2 : SP \times PB' = \text{ſin.}C : \text{ſin.}B$, quæ eſt proportio ad parabolam.

Altera proportio $PB' : S'P = \text{ſin.}S' : \text{ſin.}B'$ efficit $S'P = B'P \frac{\text{ſin.}B'}{\text{ſin.}S'}$; & impoſſibilitas rectæ $S'P$ denotatur introductione ſigni $\frac{1}{0}$, quando $S' = 0$.

3°. Conſideremus denique ſectionum ellipticarum & parabolicarum æquationes a parametro ipſarum pendentes.

Fig. 23. Sit SS' axis ellipſeos, cujus parameter ſit SA. Sit MP recta axi ordinatim applicata. Ducatur $S'A$, cui MP occurrat in N.

Per naturam ellipſis eſt $MP^2 : SP \times PS' = AS : SS' = NP : PS' = SP \times PN : SP \times PS'$; proinde $MP^2 = SP \times PN$. Ducatur per A recta axi SS' parallela, cui MP occurrat in Q; erit $MP^2 = SP(SA - NQ)$.

Quamdiu igitur curva propoſita eſt elliptica, ſeu quamdiu punctum S' datur poſitione; quadratum ordinatæ MP minus eſt rectangulo ſub parametro SA & abſciſſa SP. Atqui puncto P manente eodem; quo major eſt recta SS', eo minor fit recta $NQ \left(= SP \times \frac{SA}{SS'}\right)$: & quemadmodum nullus eſt limes magnitudinis rectæ SS', ita etiam nullus eſt limes parvitatis rectæ NQ; ſeu parameter SA limes eſt rectæ $PN (= SA - NQ)$, & rectangulum $SA \times SP$ limes eſt magnitudinis rectanguli $SP \times PN$. Ordinatæ igitur parabolæ limites ſunt ordinatarum ellipſium per eadem axis puncta ductarum; denique ipſa parabola limes eſt

elli-

ellipſium eadem parametro deſcriptarum, ad quem curvæ hæ eo propius accedunt, quo major ſit earnm axis transverſus.

Hinc intelligitur, quomodo æquatio elliptica $yy = x(p - \frac{p}{a}x)$, parabolicam $yy = px$ contineat: quatenus ſcilicet recta AS' ipſi SP non amplius occurrente; ſeu facta ipſi SS' parallela, recta PN ſit ipſi SA æqualis, & proinde NQ ſeu $\frac{p}{a}x$ ſit zero.

Quæcunque de relatione, quæ curvas inter parabolicas & ellipticas locum habet, dicta ſunt, applicari pariter debent ad relationem curvas inter parabolicas & hyperbolicas intercedentem: & quemadmodum v. gr. ordinatæ parabolæ limites ſunt ordinatarum creſcentium ellipſium eundem verticem eandemque parametrum habentium; ita etiam ordinatæ parabolæ limites ſunt ordinatarum decreſcentium hyperbolarum eundem verticem eandemque parametrum habentium. Scilicet in æquatione hyperbolæ, $yy = x(p + \frac{p}{a}x)$, parameter data p limes eſt parvitatis quantitatis mutabilis $p + \frac{p}{a}x$, & rectangulum px limes eſt parvitatis rectanguli decreſcentis $x(p + \frac{p}{a}x)$; denique rectæ in parabola axi ordinatim applicatæ limites ſunt ordinatarum decreſcentium hyperbolarum, iisdem axeos punctis reſpondentium.

Porro quæ de ſectionibus conicis dicta fuerunt, facile (mutatis mutandis) ad alias curvas ellipticas, parabolicas, & hyperbolicas ſuperiorum generum applicentur; quare diutius his immorari ſupervacaneum eſſe cenſeo.

§. 95. Alia exempla, ſententiam de vera ſymboli $\frac{1}{0}$ ſignificatione hactenus expoſitam illuſtrantia, brevius perſtringam.

1°. Sit æquatio hyperbolæ conicæ ad aſymptoton relatæ $xy = ab$; & quæratur punctum, ubi aſymptoto curva occurrit, ſeu ubi ſit $y = 0 = b \times 0$. Quoniam eſt $x = \frac{ab}{y}$; facto $y = 0 = b \times 0$, erit $x = \frac{a}{0}$; proinde occurſus mutuus curvæ hyperbolicæ & aſymptoti eſt impoſſibilis.

Æquatione $x = \frac{ab}{0}$ monemur: ſuperficie rectanguli alicujus magnitudine data, non poſſe unum ex lateribus ejus ita magnum fieri, ut alterum evaneſcat.

 2°. Sit

2°. Sit æquatio hyperbolæ conicæ ad axem transverſum relatæ, $yy = \frac{bb}{aa}(xx - aa)$; & quæratur tangens, quæ axi occurrat in centro C.

Quoniam eſt $y = \frac{b}{a}\sqrt{(xx-aa)}$; fit $\frac{dy}{dx} = \frac{b}{a} \times \frac{x}{\sqrt{(xx-aa)}}$, $\frac{dx}{dy} = \frac{a}{b} \times \frac{\sqrt{(xx-aa)}}{x}$, & $y\frac{dx}{dy} = \frac{xx-aa}{x} = x - \frac{aa}{x}$, unde diſtantia centri a puncto, ubi tangens axi occurrit, eſt $\frac{aa}{x}$: quare diſtantia hac magnitudine data d, fit $\frac{aa}{x} = d$; & $x = \frac{aa}{d}$. Poſita autem $d = 0 = a \times 0$, fit $x = \frac{a}{0}$, & proinde impoſſibilis. Nulla igitur eſt tangens hyperbolæ, quæ axi in centro occurrat; ſed punctum, ubi tangens axi occurrit, eo propius ad centrum accedit, quo punctum, ex quo tangens ducitur, a centro longius removetur.

3°. Sit parabola conica, cujus æquatio eſt $y = \sqrt{2px}$; & quæratur punctum parabolæ, ex quo ducta tangens fiat axi parallela.

Quoniam $y = \sqrt{2px}$; fit $\frac{dy}{dx} = \frac{p}{\sqrt{2px}} = \frac{p}{y}$. Jam vero (§. 41.) $\frac{dy}{dx}$ eſt tangens trigonometrica (quæ dicatur t) anguli, ſub quo tangens parabolæ axi occurrit. Eſt ideo $\frac{p}{y} = t$; & $y = \frac{p}{t}$. Fiat autem tangens axi parallela, & proinde nullum faciat angulum cum axe: erit etiam $t = 0$, & $y = \frac{p}{0}$; quod impoſſibilitatem arguit ducendi rectam axi parabolæ parallelam, quæ hanc curvam tangat.

§. 96. Ne quis autem hactenus dicta male interpretetur, quaſi ſigna $\frac{1}{0}$, $(\frac{1}{0})^n$ ab algebriſtis introducta ex matheſi proſcribere velim. Quemadmodum alia impoſſibilitatis ſigna $\sqrt{-1}$, $\sqrt[2n]{-1}$, a mathematicis uſurpata, plurimas inveſtigationes mirifice juvant, quæ abſque illorum ſubſidio (pro intellectus humani debilitate) aut impoſſibiles fuiſſent aut longe difficiliores; ita etiam hæc impoſſibilitatis ſigna, $\frac{1}{0}$, $(\frac{1}{0})^n$, adminicula eſſe poſſunt & inſtrumenta, quibus intellectus humanus juvatur, ut ad inveſtigationes reales poſſit eniti. Dicam ideo cum omnibus algebriſtis: generalem ſectionum conicarum (v. gr.) æquationem

nem eſſe $yy = x(p \pm \frac{p}{a}x)$; quæ caſu, quo $a = \frac{1}{0} = \infty$, ſit pro parabola $yy = px$. Spero autem, tirones verum illationis hujus ſenſum & nexum ejus cum theoria limitum ex præcedentibus habituros eſſe perſpectum; nec ob introductionem forte neceſſariam ſymbolorum $\frac{1}{0}$, $(\frac{1}{0})^n$, ſeu ∞, ∞^n procliviores fore ad magnitudines reales ſignis his ſubſternendas, quam proni fuerint ad admittendam quantitatum (quas vocant) imaginariarum exiſtentiam, quarum ſymbola in calculos quaſi neceſſario irrepunt, aut majoris ſaltem facilitatis cauſa adhibentur.

Superfluum autem cenſeo, oſtendendis introductionis ſignorum horum commodis immorari, quæ frequens recentiorum mathematicorum uſus abunde comprobat.

§. 97. Sententiam pluribus hic expoſitam, quod ſcilicet ſymbolum $\frac{1}{0}$ ſignum ſit impoſſibilitatis, breviter jam attigeram in diſſertatione: *Expoſition elementaire* &c. inprimis Cap. XI. pag. 158. 159. 165. Eandem profitetur ill. Kæstner in diſſertatione inſcripta *De translatis in dictione geometrarum*, his verbis: „ Ex vulgato tang. $n = \frac{\text{ſin.}\, n}{\text{coſ.}\, n}$ eruitur tangens anguli recti infinita — — — „ Revera autem angulus rectus nec coſinum habet, nec tangentem. Coſinum „ non habere utique dico, cum dico, ejus coſinum eſſe = o; non enim habet, „ qui nihil habet. Hinc facile intelligitur, tangentem etiam habere non poſſe. „ Tangentem anguli recti infinitam eſſe nihil aliud dicit, quam anguli ad rectum creſcentis tangentem creſcere ultra omnes limites, ita ut coſinus infra „ omnes limites decreſcit. Eo momento, quo angulus fit rectus, ceſſant notio „ tangentis & coſinus; hoc diſcrimine, quod in eum ſtatum, ut coſinus notio „ ceſſet, res deducta ſit perpetuis decrementis coſinus; in eum vero, ut ceſſet „ notio tangentis, perpetuis incrementis tangentis.„ Ac nuperrime Celeb. Abbas Caluso in *novis Commentariis Academiæ Turinenſis* T. II. idem tradit judicium his aliisque verbis: „Que l'on demande *t* tang. de *z*. Dans ce probleme „ ce n'eſt point la poſition de *t*, c'eſt la grandeur de *t*, que l'on veut. Cette „ grandeur eſt la diſtance du point d'attouchement à l'interſection de la tangente & de la ſecante, qui fait l'angle *z* avec le raïon perpendiculaire à la „ tan-

„ tangente. Ainſi, le point d'attouchement demeurant, à meſure que j'en eloi-
„ gne celui de l'interſection, je fais croitre t & z. Mais je ne puis jamais eloi-
„ gner l'interſection pour avoir $z = 90°$, parceque l'interſection de deux paral-
„ leles eſt impoſſible. Or on ne peut concevoir de diſtance d'une interſection,
„ que l'on conçoit impoſſible. Il faut donc concevoir t impoſſible, lorsque
„ $z = 90°$.

„ Il faut diſtinguer trois cas d'impoſſibilité.

„ 1°. Celui que nous venons de remarquer dans t, impoſſible parce qu'on
„ ne peut aſſés eloigner les extrèmes, qui doivent terminer la grandeur.

„ 2°. Celui où elle eſt impoſſible, parce qu'on ne peut aſſés les aprocher,
„ dont il ſuffira de donner pour exemple la cotangente de 90°.

„ 3°. Celui où un des extrèmes devroit être en même tems des deux co-
„ tés oppoſés de l'autre; c'eſt le cas des imaginaires, dont l'impoſſibilité a tou-
„ jours eté reconnuë. „

§. 98. Admiſſo, ſymbolum $\frac{1}{0}$ ſignum eſſe impoſſibilis; mirum non eſt, mathematicorum conatus rem ſigno huic reſpondentem exprimendi eſſe inanes. Methodos inter, quibus ſigni hujus indolem definire annixi ſunt, ſequens notari meretur reductio. Poſito $0 = 1 - 1$, & $\frac{1}{0} = \frac{1}{1-1}$, in ſeriem convertatur expreſſio $\frac{1}{1-1}$; erit $\frac{1}{1-1} = 1 + 1 + 1 + 1 + 1 + \ldots$ Quæ ſeries cum ſine limite poſſit continuari, inferunt: ſignum $\frac{1}{0}$ ſymbolum eſſe quantitatis realis, ſeriei nempe infinitæ terminis invicem æqualibus conſtantis. Circa hanc reductionem hæc mihi obſervanda videntur.

1°. Quoniam 0 eſt differentia (ſic dicta) duarum quantitatum invicem æqualium; eodem omnino jure fiet $0 = n - n$, & $\frac{1}{0} = \frac{1}{n-n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots$ Jam vero ponatur $n = \frac{1}{0}$: erit quoque $\frac{1}{0} = 0 + 0 + 0 + 0 + \ldots$; quæ ſeries (ſi nihilorum congeriem ita nuncupare licet) quantitatem infinitam dictam $\frac{1}{0}$ nunquam efficere poteſt.

2°. Sem-

2°. Semper eſt $\frac{1}{a-b}=\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\ldots+\frac{b^{n-2}}{a^{n-1}}+\frac{b^{n-1}}{a^n}+\frac{\frac{b^n}{a^n}}{a-b}$. Proinde ſeries ex evolutione fractionis $\frac{1}{a-b}$ orta ad valorem hujus fractionis eo tantum caſu propius continue accedit, quo $b < a$; ita ut ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ eo minus fiat, quo major eſt terminorum evolutorum numerus. Contra, quando $b > a$, ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ eo majus fit, quo major eſt terminorum evolutorum numerus; proindeque ſeries evoluta a fractione $\frac{1}{a-b}$ eo magis diſcrepat, quo major eſt terminorum evolutorum numerus. Caſu autem, quo $a=b$, ſupplementum $\frac{\frac{b^n}{a^n}}{a-b}$ idem ſemper manet; & proinde ſeries evoluta neque accedit ad valorem fractionis primariæ $\frac{1}{a-b}$, neque ab illo recedit. Quin cum hoc caſu, quo $a=b$, ſupplementum $\frac{\frac{a^n}{b^n}}{a-b}$ ſeriei, quousque libeat continuatæ, fit ipſi fractioni $\frac{1}{a-a}$ in ſeriem explicandæ æqualis; perpetuo hoc ipſiusmet recurſu docemur, converſionem fractionis $\frac{1}{a-a}$ in ſeriem ineptam eſſe ad determinandam rem, quæ ipſi reſpondeat.

Quoniam ideo $\frac{1}{0}=1+1+1+\ldots+\frac{1}{0}$; cur non liceret concludere ſummam $1+1+1+\ldots=0$? Sed revera nihil aliud ex hac æquatione concludi poteſt, niſi quod $1=0+0+0+\ldots+0+1$, ſeu $1=1$.

Scholium. Fractionis $\frac{1}{a-b}$ in ſeriem converſio caſu, quo $b > a$ (& proinde $\frac{1}{a-b}=-\frac{1}{b-a}$) mathematicos quosdam eo deduxit, ut quantitates (quas vocant) negativas eſſe plusquam infinitas (a) pronuntiarent. Cum fractionis $\frac{1}{a-b}$ in

(a) Ipſe WALLISIUS, qui ad matheſeos progreſſus tantopere contulit, nonniſi timide hanc vocem primus (quod ſciam) emiſit. De rationibus plus quam infinitis locutus addit: *ſi id ſine ſolecismo dici poſſit.* (Vid. *Arithmetica infinitorum.* Prop. CI. Scholium.)

in ſeriem converſio caſu, quo $b > a$, nihil doceat, quod ad valorem iſtius fra-ctionis, niſi ratio habeatur ſupplementi $\frac{\frac{b^n}{a^n}}{a-b}$ eo majoris, quo major eſt terminorum evolutorum numerus; patet, inſolens illud effatum omni hoc reſpectu fundamento carere.

Caſu, quo $b > a$, fractio $\frac{1}{a-b}$ eſt $-\frac{1}{b-a}$, quæ in ſeriem convergentem evoluta fit $-\left(\frac{1}{b}+\frac{a}{bb}+\frac{a^2}{b^3}+\frac{a^3}{b^4}+\frac{a^4}{b^5}+\ldots\right)$; & abſurdum foret dicere, ſeriem hanc eandem eſſe cum altera $\frac{1}{a}+\frac{b}{aa}+\frac{b^2}{a^3}+\frac{b^3}{a^4}+\frac{b^4}{a^5}+\ldots$.

Si ad exemplum familiare, quo hoc caput exorſi ſumus (§. 89.), redeamus; luculenter apparebit, quantum hæc ſententia a ſenſu communi abhorreat. Expreſſio $D \times \frac{V}{V-v}$ caſu, quo $v > V$, fit $-D \times \frac{V}{v-V}$; quæ indicat, punctnm occurſus duorum viatorum jacere ad partes oppoſitas iis, verſus quas viatores progrediuntur. Neque ullo modo concipi poteſt, ſpatium ab illis percurrendum majus eſſe ſpatio (ficto) $D \times \frac{V}{V-v} = D \times \frac{1}{0}$, quod velocitatum æqualitati reſpondeat.

Nec magis ſolidum eſt, quod nonnulli mathematici dicunt: hyperbolas eſſe parabolas plus quam infinitas. Parabola curva eſt ſui generis, ad quam, tanquam limitem, propius propiusque accedere poſſunt reliquæ ſectiones conicæ. Et quemadmodum polygona circulo inſcripta aut circumſcripta ſemper a circulo differunt; ita etiam ellipſes & hyperbolæ a parabola ſemper diſcrepant.

Huc etiam pertinent, quæ (§. 71.) de ſignorum operationum ad quantitates ipſas translatione obſervavi.

§. 99. Admiſſo, ſymbolum $\frac{1}{0}$ ſignum eſſe impoſſibilitatis; facile intelligemus, quid ſibi velint ſigna $\frac{1}{0^n}$, ∞^n, a mathematicis uſurpata, quibus varios infinitorum ordines diſtinguere autumarunt. Quemadmodum omnes quantitates imaginariæ, utut exponentibus diverſis affectæ, ad ſignum unicum impoſſibilitatis $\sqrt{-1}$ reducuntur; ita etiam omnia ſymbola $\frac{1}{0^n}$, ∞^n, unicam indicare im-

poſſi-

poſſibilitatis ideam aſſerere non dubito. Id quod uno aut altero exemplo liquebit.

Sit æquatio ellipſium $y^m = \frac{b^m}{a^m}(a^m - x^m)$; $y = \frac{b}{a}(a^m - x^m)^{\frac{1}{m}}$ Fig. 11.

$$\text{unde ſubtang.} \left(= y\frac{dx}{dy}\right) = \frac{a^m}{x^{m-1}} - x = PT.$$

Quo minor eſt x, eo major eſt fractio $\frac{a^m}{x^{m-1}}$; eademque tanto rapidius creſcit, quo major eſt exponens m. Ipſo autem momento, quo fit $x = 0$, ſymbolum $\frac{a^m}{0^{m-1}}$ non majorem indicat occurſus mutui tangentis & axis impoſſibilitatem, quam ſignum ſimplex impoſſibilitatis $\frac{1}{0}$, quod obtinet, quando $m = 2$, uti in ellipſi conica. In ellipſibus variorum ordinum ſubtangentes inæqualiter tendunt verſus impoſſibilitatem, quamdiu actu exiſtunt; ſed unico modo eam attingunt, aut in ea perſiſtunt. Recta ex puncto poſitione dato ducta ſecundum innumeras leges ad eum tendere poteſt ſitum, quo fiat rectæ poſitione datæ parallela; ſed ſitus hic eſt unicus, quicunque ſit modus, quo ad illum pervenerit.

Sit æquatio hyperbolarum ad axem transverſum relatarum $y^m = \frac{b^m}{a^m}(x^m - a^m)$; ſubtang. $\left(= y\frac{dx}{dy}\right) = x - \frac{a^m}{x^{m-1}}$. Puncti igitur, ubi tangens axi occurrit, a centro hyperbolarum diſtantia eſt $a\frac{a^{m-1}}{x^{m-1}}$: & ſi requiratur, ut diſtantia hæc evaneſcat; erit $a\frac{a^{m-1}}{x^{m-1}} = 0$, ſeu $x^{m-1} = a^{m-1} \times \frac{a}{0}$, $x = a\sqrt[m-1]{\frac{a}{0}}$. Quo ſymbolo monemur, impoſſibile eſſe, ut tangentes hyperbolarum cujuscunque ordinis axi in centro occurrant; neque impoſſibilitas hæc major minorve eſſe dici poteſt pro vario ordine hyperbolarum.

Quæcunque dixi de ſymboli $\frac{1}{1-1}\left(= \frac{1}{0}\right)$ in ſeriem converſione, a fortiori debent ad ſeries ex ſymboli $\frac{1}{(1-1)^n}\left(= \frac{1}{0^n}\right)$ in ſeries converſione ortas applicari, qua varios infinitorum ordines declarare fuit tentatum.

Omitto alia argumenta ex contemplatione ſerierum & ſpatiorum curvilineorum deducta: quæ plerumque contradictoria nituntur ſuppoſitione, dari aliquam quantitatem infinitam; & de quibus, occaſione data, ſtrictim dicere ſufficiet.

Caput Decimum.

De quadratura curvarum.

§. 100.

Curva quævis referatur ad aliquem axem per rectas huic axi ordinatim applicatas, & eidem v. gr. perpendiculares. (*a*) Superficies curvæ denotet aream trianguli mixtilinei, quod arcu curvæ & abſciſſa atque ordinatim applicata axis terminatur; vel aream quadrilateri mixtilinei, arcu curvæ, abſciſſa axis, & duabus ejusdem ordinatis comprehenſi. Sit rectangulum, cujus unum latus detur magnitudine, & cujus latus alterum creſcat uti abſciſſa axis. Dico: rationem differentialem ſuperficierum curvæ & rectanguli æqualem eſſe rationi rectæ axi ordinatim applicatæ, quæ eſt baſis ſuperficiei curvæ, ad rectam magnitudine datam.

Fig. 24. Eſto SMM' curva aliqua, ad axem SPP' relata per rectas MP, $M'P'$ huic ordinatim & perpendiculariter applicatas. Ad punctum S conſtituatur recta indefinita axi perpendicularis. Fiant rectangula $SARP$, $SAR'P'$, quorum latus commune SA æquale ſit rectæ magnitudine datæ, & altera latera ſint abſciſſæ axis SP, SP'. Dico, rationem differentialem ſuperficiei curvæ & rectanguli æqualem eſſe rationi $MP : SA$.

Etenim dum abſciſſa SP creſcit quantitate PP', mutationes ſimultaneæ ſuperficierum curvæ & rectanguli ſunt ſpatia $MPP'M'$ & $RPP'R'$. Compleantur rectangula $PP'M'm'$, $PP'mM$, quorum prius eſt curvæ circumſcriptum, poſterius inſcriptum (ad normam §. 1. Ex. 3.).

Ratio mutationum ſimultanearum ſuperficiei curvæ SMP & rectanguli SR minor eſt ratione rectangulorum PM', PR', ſeu ratione $PM' : SA$; major autem ratione rectangulorum Pm, PR', ſeu ratione $PM : SA$. Quoniam autem ratio æqualitatis limes eſt rationis rectarum $M'P'$, MP, ſeu rectangulorum PM', Pm; ratio rectangulorum PM', PR' minor fieri poteſt quacunque ratione propoſita, quæ

(*a*) Si coordinatarum angulus non ſit rectus; quæcunque de rectangulis dicentur, transferenda ſunt ad parallelogramma æqui-angula, quorum anguli ſunt angulo coordinatarum æquales.

quæ major ſit ratione $PM : SA$; & a fortiori ratio ſpatii $PMM'P'$ ad rectangulum PR' minor fieri poteſt quacunque ratione propoſita, quæ major eſt ratione $PM : SA$. Proinde ratio $PM : SA$ limes eſt rationis mutationum ſimultanearum ſpatiorum SMP & $SARP$, ſeu eſt ratio differentialis horum ſpatiorum.

Sit ideo $SP = x$, $MP = y$, $SA = a$, & ſuperficies curvæ dicatur S; erit $\lim.\frac{\Delta S}{a\Delta x} = \frac{y}{a}$, ſeu $\lim.\frac{\Delta S}{\Delta x} = y$, & $\frac{dS}{dx} = y$.

Data igitur (ex æquatione curvæ) relatione x & y, determinatio propoſita ſuperficiei curvæ ad calculum integralem reducta eſt.

Exemplum. Sit $a^{m-1}y = x^m$; erit $\frac{dS}{dx} = y = \frac{x^m}{a^{m-1}}$, &

$S = C + \frac{1}{m+1}\cdot\frac{x^{m+1}}{a^{m-1}} = C + \frac{1}{m+1}xy$.

1. *Caſus.* Sit m numerus poſitivus, & initium curvæ ſit in vertice axis. Curva propoſita eſt parabolica. Et quoniam ſuperficies (per hyp.) evaneſcit facto $x = 0$, eſt $C = 0$; unde $S = \frac{1}{m+1}xy$. Area igitur ſemi-ſegmenti parabolici, cujus æquatio eſt $y = \frac{x^m}{a^{m-1}}$, eſt ad rectangulum circumſcriptum, uti 1 : $m+1$.

2. *Caſus.* Sit m numerus negativus, ſeu ſit $yx^m = a^{m+1}$, quæ eſt æquatio hyperbolarum ad aſymptotos relatarum. Quoniam $\frac{dS}{dx} = y = a^{m+1}x^{-m}$; fit $S = C - \frac{1}{m-1}a^{m+1}x^{-m+1} = C - \frac{1}{m-1}xy$. Ut caſus hujus examen fiat facilius, ſit $AB = b$ ordinata, cui reſpondet abſciſſa $SB = a$, a cujus extremo B ſumatur initium abſciſſarum; erit ideo $y(a+x)^m = ba^m$; $\frac{dS}{dx} = a^m b(a+x)^{-m}$,

$S = C - \frac{1}{m-1}a^m b(a+x)^{-m+1} = C - \frac{1}{m-1}\cdot\frac{a^m b}{(a+x)^{m-1}} = C - \frac{1}{m-1}y(a+x)$. Jam Fig. 25.
vero ſuperficies propoſita $ABPM$ evaneſcit, facto $BP = x = 0$, & $y = AB = b$; proinde $C = + \frac{1}{m-1}ab$, & $S = \frac{1}{m-1}(ab - y(a+x))$.

1°. Sit $m > 1$, $y(a+x) = \frac{a^m b}{(a+x)^{m-1}} = ab\left(\frac{a}{a+x}\right)^{m-1}$. Quoniam autem $a+x$ ſeu SP major fieri poteſt quacunque quantitate propoſita; contra $\frac{a}{a+x}$

 &

& $\left(\frac{a}{a+x}\right)^{m-1}$ minor fieri poteſt quacunque quantitate propoſita: proinde ab limes eſt quantitatis creſcentis $ab-y(a+x)$; adeoque ſpatium $\frac{1}{m-1}ABSD$ limes eſt ſpatii aſymptotici creſcentis $ABPM$, quod æquare, nedum ſuperare nequit, utcunque magna ſumatur abſciſſa BP.

In eodem caſu fiat x negativa; unde $y(a-x)^m = a^m b$, & $y = a^m b(a-x)^{-m}$.

Erit $S = \frac{1}{m-1}a^m b\left((a-x)^{-m+1} - C\right)$,

$$= \frac{1}{m-1}a^m b\left((a-x)^{-m+1} - a^{-m+1}\right)$$

$$= \frac{1}{m-1}a^m b\left(\frac{1}{\overline{a-x}^{m-1}} - \frac{1}{a^{m-1}}\right)$$

$$= \frac{1}{m-1}\quad\left(y(a-x) - ab\right).$$

Quoniam autem nullus eſt limes parvitatis quantitatis $a-x$, inde ab a ad zero usque decreſcentis: contra nullus eſt limes magnitudinis quantitatum creſcentium $\frac{a}{a-x}$, $\frac{a^{m-1}}{(a-x)^{m-1}}$; & proinde nullus eſt limes magnitudinis ſpatii aſymptotici ad alteras partes ordinatæ AB.

Hoc caſu poſito $x = a$, fit $S = \frac{1}{m-1}ab.\frac{1}{0^{m-1}}$; quod ſymbolum indicat, tam contradictionem ſuppoſitionis, fieri poſſe $x = a$, ſeu hyperbolam aſymptoto occurrere, quam impoſſibilitatem ſpatii aſymptotici limitem determinandi.

2°. Sit $m < 1$: erit $y(a+x) = \frac{a^m b}{(a+x)^{m-1}} = ab \times \left(\frac{a+x}{a}\right)^{1-m}$, & $S = \frac{1}{1-m}ab\left(\left(\frac{a+x}{a}\right)^{1-m} - 1\right) = \frac{1}{1-m}(y(a+x) - ab)$. Quoniam autem nullus eſt limes magnitudinis ipſius $a+x$, nullus etiam eſt limes magnitudinis quantitatum $\frac{a+x}{a}$, $\left(\frac{a+x}{a}\right)^{1-m}$; & proinde, creſcente abſciſſa BP, ſpatium aſymptoticum creſcit, & majus fieri poteſt quocunque ſpatio dato.

Sit autem x negativa. Quoniam $y(a-x)^m = a^m b$, erit ut prius $S = \frac{1}{1-m}(ab - y(a-x)) = \frac{1}{1-m}ab\left(1 - \left(\frac{a-x}{a}\right)^{1-m}\right)$: & quoniam, creſcente x a zero

inde

inde usque ad a, nullus eſt limes parvitatis quantitatum $\frac{a-x}{a}$, $\left(\frac{a-x}{a}\right)^{1-m}$; ſpatium aſymptoticum ex altera parte ordinatæ AB obtinet limitem magnitudinis, nempe $\frac{1}{1-m}ab$, ſeu $\frac{1}{1-m}AB \times SB$.

Hoc caſu eſt etiam $S = \frac{1}{1-m}ab\left(\left(\frac{b}{y}\right)^{\frac{1-m}{m}} - 1\right)$. Facto $y = 0$, fit $S = \frac{1}{1-m}ab\left(\left(\frac{b}{0}\right)^{\frac{1-m}{m}} - 1\right)$; quod indicat tam contradictionem ſuppoſitionis, fieri poſſe $y = 0$, quam impoſſibilitatem ſpatii aſymptotici limitem determinandi.

3°. Sit $m = 1$. Hoc caſu $\frac{dS}{dx} = y = \frac{ab}{a+x}$, quæ eſt æquatio differentialis logarithmica. Proinde $S = ab(C + \log. a + x)$. Et quoniam S evaneſcit, facto $x = 0$; fit $C = -\log. a$, & $S = ab \log. \frac{a+x}{a}$. Proinde ſpatia hyerbolica $ABPM$ creſcunt ut logarithmi rationum $SP : SB$ abſciſſarum.

Geometricam proprietatis hujus hyperbolæ conicæ demonſtrationem ad calcem hujus capitis differre ſatius mihi eſſe videtur, ne principalium propoſitionum ſeries nimium abrumpatur.

Quando $m \gtrless 1$; alterutrum ſpatiorum aſymptoticorum, ordinata aliqua AB ſeparatorum, limitem habet magnitudinis. Caſu autem, quo $m = 1$, utrumque horum ſpatiorum majus fieri poteſt quocunque ſpatio dato.

§. 101. Quoniam curvæ parabolicæ & hyperbolicæ magni ſunt in theoria curvarum momenti; haud abs re erit earum quadraturam paulo aliter inveſtigare, & ad prima limitum rationum principia reducere.

1°. Curva quadranda ſit parabola SMM', cujus æquatio $a^{m-1}y = x^m$. Ad verticem S ducatur tangens SA; ducatur etiam ad M tangens curvæ, quæ axi SP occurrat in T. Ducatur ordinata $M'P'$; per M & M' ducantur ipſi SA perpendiculares MQ, $M'Q'$, quæ ipſis $M'P'$, MP (productis, ſi neceſſe fuerit) in m & m' occurrant. Fig. 24.

Rect.

$$\text{Rect.}\,MPP'm : \text{rect.}\,MQQ'm' = MP \times Mm : MQ \times M'm.$$

$$\begin{array}{lllllll} \text{Sed} & \text{lim.}\,Mm & : & M'm & = & PT : MP & (\text{§. 40.}) \\ \text{et} & MP & : & MQ & = & MP : MQ & \\ \text{Ergo} & \text{lim.}(MP \times Mm & : & MQ \times M'm) & = & PT : MQ & (\text{§. 14.}) \\ & & & & = & PT : SP & \\ & & & & = & 1 : m & (\text{§. 42.}) \end{array}$$

Proinde $\text{lim.ſ.rect.}\,MPP'm : \text{lim.ſ.rect.}\,MQQ'm' = 1 : m$ (§. 15.)

Sed ſpatium parabolicum exterius $SMM'Q'$ limes eſt ſummæ rectangulorum huic inſcriptorum $MQQ'm'$, & ſpatium parabolicum internum $SMM'P'$ limes eſt rectangulorum huic pariter inſcriptorum $MPP'm$; proinde (§. 4.)

$$\begin{array}{llllll} & SMM'P' & : & SMM'Q' & = & 1 : m \\ \text{ſeu} & SMP & : & SMQ & = & 1 : m \\ \text{hinc} & SPMQ & : & SMP & = & m+1 : 1 \\ & SPMQ & : & SMQ & = & m+1 : m. \end{array}$$

2°. Curva quadranda ſit hyperbola, cujus aſymptotæ SP, SQ, & cujus æquatio eſt $y(a+x)^m = a^m b$. Per M ducatur tangens, quæ aſymptotis in T & T' occurrat. Sint MP, MQ, $M'P'$, $M'Q'$ rectæ aſymptotis ordinatim applicatæ. Rectæ MP, $M'Q'$ ſibi mutuo occurrant in m', & rectæ MQ, $M'P'$ ſibi mutuo occurrant in m.

Fig. 25.

$$\begin{array}{lllllll} & \text{Rect.}\,MPP'm & : & \text{rect.}\,mQQ'M' & = & MP \times M'm' : MQ \times Mm'. & \\ \text{Sed} & \text{lim.}\,M'm' & : & Mm' & = & PT : MP & (\text{§. 40.}) \\ & \text{lim.}\,MP & : & mQ & = & MP : MQ & \\ \text{ergo} & \text{lim.}\,MP \times M'm' & : & mQ \times Mm' & = & PT : MQ & (\text{§. 14.}) \\ & & & & = & PT : SP & \\ & & & & = & 1 : m & (\text{§. 42.}) \end{array}$$

Ergo $\text{lim.}(\text{ſ.rect.}\,PMmP' : \text{ſ.rect.}\,mQQ'M') = 1 : m$ (§. 15.)

Sed ſpatium hyperbolicum $ABP'M'$ limes eſt ſummæ rectangulorum $PMmP'$ (§. 1.), & ſpatium hyperbolicum $ADQ'M'$ limes eſt ſummæ rectangulorum $mQQ'M'$. Ergo (§. 4.) $ABP'M' : ADQ'M' = 1 : m$

atque etiam $ABPM : ADQM = 1 : m.$

1°. Sit

1°. Sit $m < 1$; erit $ABPM > ADQM$; & $ABPM - ADQM = MRBP - ARQD = MQSP - ABSD$.

$$\begin{aligned} \text{Quoniam}\quad ABPM &: ADQM = 1 : m \\ ABPM &: MQSP - ABSD = 1 : 1 - m \\ \&\quad ADQM &: MQSP - ABSD = m : 1 - m. \end{aligned}$$

2°. Sit $m > 1$; erit $ABPM < ADQM$, & $ADQM - ABPM = ARQD - MRSP = ABSD - MQSP$.

$$\begin{aligned} \text{Quoniam}\quad ABPM &: ADQM = 1 : m \\ ABPM &: ABSD - MQSP = 1 : m - 1 \\ \&\quad ADQM &: ABSD - MQSP = m : m - 1. \end{aligned}$$

3°. Sit $m = 1$. Nil aliud inde concludi poteſt, niſi quod $ABPM = ADQM$.

§. 102. Quadratura parabolarum & hyperbolarum viam ſternit ad quadraturam plurimarum curvarum, quarum ſuperficies ab illarum ſuperficiebus pendent.

Sit exempli gratia $bby = xx(a-x)$

erit $\dfrac{dS}{dx} = y = \dfrac{axx - x^3}{bb}$. $S = \dfrac{\frac{1}{3}ax^3 - \frac{1}{4}x^4}{bb} = \dfrac{x^3(\frac{1}{3}a - \frac{1}{4}x)}{bb}$.

Sit $S = 0$, quando $x = 0$, & ſumatur ſuperficies reſpondens abſciſſæ $x = a$; erit $S = \frac{1}{12}\dfrac{a^4}{bb}$.

Sit in genere

$b^{m+n-1}y = x^m(a-x)^n$; erit

$$\frac{dS}{dx} = y = \frac{x^m(a-x)^n}{b^{m+n-1}}$$

$$= \frac{x^m}{b^{m+n-1}}\left(a^n - \frac{n}{1}a^{n-1}x + \frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}x^2 - \frac{n}{1}\cdots\frac{n-2}{3}a^{n-3}x^3 + \frac{n}{1}\cdots\frac{n-3}{4}a^{n-4}x^4 - \ldots\right)$$

unde

$$S = \frac{1}{b^{m+n-1}}\left(\frac{1}{m+1}a^n x^{m+1} - \frac{n}{1}\cdot\frac{1}{m+2}a^{n-1}x^{m+2} + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{1}{m+3}a^{n-2}x^{m+3} - \frac{n}{1}\cdots\frac{n-2}{3}\cdot\frac{1}{m+4}a^{n-3}x^{m+4} + \ldots\right)$$

Quæ ſeries abrumpitur, ſi fuerit n numerus integer & poſitivus.

Curva propoſita ſit circulus, cujus æquatio eſt $yy = rr - xx$; $\dfrac{dS}{dx} = y = \sqrt{rr - xx}$. Formula integranda in ſeriem convertatur; erit

$\dfrac{dS}{dx}$

$$\frac{dS}{dx} = r - \frac{1}{2}\frac{xx}{r} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\frac{x^4}{r^3} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\frac{x^6}{r^5} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{\frac{5}{2}}{4}\frac{x^8}{r^7} - \frac{1}{2}\ldots\frac{\frac{7}{2}}{5}\cdot\frac{x^{10}}{r^9} - \ldots\ldots$$

$$S = rx - \frac{1}{2}\cdot\frac{1}{3}\frac{x^3}{r} - \frac{\frac{1}{2}\cdot\frac{1}{2}}{1.2}\cdot\frac{1}{5}\frac{x^5}{r^3} - \frac{\frac{1}{2}}{1}\ldots\frac{\frac{3}{2}}{3}\cdot\frac{1}{7}\frac{x^7}{r^5} - \frac{\frac{1}{2}}{1}\ldots\frac{\frac{5}{2}}{4}\cdot\frac{1}{9}\frac{x^9}{r^7} - \frac{\frac{1}{2}}{1}\ldots\frac{\frac{7}{2}}{5}\cdot\frac{1}{11}\cdot\frac{x^{11}}{r^9} - \ldots$$

Sit $x = r$, & proinde S quadrans circuli; erit

$$S = rr(1 - \tfrac{1}{2}\cdot\tfrac{1}{3} - \tfrac{1}{8}\cdot\tfrac{1}{5} - \tfrac{1}{16}\cdot\tfrac{1}{7} - \tfrac{5}{128}\cdot\tfrac{1}{9} - \tfrac{35}{1280}\cdot\tfrac{1}{11} - \ldots.)$$

Si curva propoſita eſſet ellipſis, cujus æquatio eſt $yy = \frac{bb}{aa}(aa - xx)$; formula eadem eſt faciendo $rr = ab$.

Curva propoſita ſit hyperbola conica ad axem transverſum relata, cujus æquatio eſt $yy = \frac{bb}{aa}(xx - aa)$.

$$\frac{dS}{dx} = y = \frac{b}{a}\sqrt{(xx\text{-}aa)} = \frac{b}{a}\left(x - \frac{1}{2}\frac{aa}{x} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\frac{a^4}{x^3} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\frac{a^6}{x^5} - \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{\frac{5}{2}}{4}\frac{a^8}{x^7} - \ldots.\right)$$

$$S = C + \frac{b}{a}\left(\tfrac{1}{2}xx - \tfrac{1}{2}aa\log.x + \frac{1}{2}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{1}{2}\frac{a^4}{x^2} + \frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{1}{4}\cdot\frac{a^6}{x^4} + \frac{\frac{1}{2}}{1}\cdots\frac{\frac{5}{2}}{4}\cdot\frac{1}{6}\cdot\frac{a^8}{x^6} + \ldots.\right)$$

Sit $S = 0$, quando $x = a$;

$$S = \frac{b}{a}\left((\tfrac{1}{2}xx - aa) - \tfrac{1}{2}aa\log.\frac{x}{a} + \tfrac{1}{8}\cdot\tfrac{1}{2}\frac{aa(xx-aa)}{x^2} + \tfrac{1}{16}\cdot\tfrac{1}{4}\cdot\frac{aa(x^4-a^4)}{x^4} + \tfrac{5}{128}\cdot\tfrac{1}{6}\cdot\frac{aa(x^6-a^6)}{x^6} + \ldots\right).$$

Obſervatio 1. Hyperbolæ conicæ ad aſymptoton relatæ quadraturam ad logarithmos reduci ſuperius vidimus (§. 100.): ideo etiam quadratura curvæ hujus ad logarithmos reduci poterit, ad quemcunque axem curva referatur. Reipſa ex ratione differentiali $\frac{dS}{dx} = \frac{b}{a}\sqrt{xx - aa}$

conſequitur etiam $S = \frac{1}{2}ab\log.\frac{x - \sqrt{xx - aa}}{a} + \frac{1}{2}\frac{b}{a}x\sqrt{xx - aa}$

$= \frac{1}{2}ab\log.\frac{a}{x + \sqrt{xx - aa}} + \frac{b}{a}x\sqrt{xx - aa}.$

Si vero hyperbola ad axem ſecundum refertur, ut æquatio ſit $yy = \frac{aa}{bb}(xx+bb)$, & $\frac{dS}{dx} = y = \frac{a}{b}\sqrt{(xx+bb)}$; pariter eſt

$$S = \tfrac{1}{2}ab\log.\frac{b}{\sqrt{(xx+bb)} - x} + \frac{1}{2}\frac{a}{b}x\sqrt{xx+bb}.$$

Obſervatio 2. Ad duas formulas $\frac{dS}{dx} = \sqrt{aa-xx}$; $\frac{dS}{dx} = \sqrt{xx \pm aa}$, reducuntur quam plurimæ aliæ formulæ, quæ proinde in terminis finitis integrabiles cenſentur; cum in promtu ſint quadratura (appropinquata) circuli, & tabulæ logarithmicæ.

Sic exiſtente *m* numero integro formulæ $\frac{dS}{dx} = x^m\sqrt{(aa-xx)}$, $\frac{dS}{dx} = x^m\sqrt{(xx \pm aa)}$ aut ad duas formulas præcedentes reducuntur; aut ad formulas $\frac{x}{\sqrt{aa-xx}}$, & $\frac{x}{\sqrt{xx \pm aa}}$, quæ ſunt immediate integrabiles.

Sit æquatio curvæ logarithmicæ $\frac{b}{y} = e^{\frac{x}{t}}$; erit $\frac{dx}{dy} = -\frac{t}{y}$.

$$\text{Sed } \frac{dS}{dx} = y$$

$$\text{ergo } \frac{dS}{dx} = -t, \ \& \ S = C - ty.$$

Sit $S = 0$, quando $y = b$; erit $C = bt$, & $S = t(b-y)$. Quoniam y minor fieri poteſt quacunque quantitate propoſita, limes ſpatii aſymptotici curvæ logarithmicæ eſt bt. Quæ formulæ immediate poſſunt ex primis limitum principiis demonſtrari.

Sit æquatio generalis cycloidum $y = \sqrt{(2rx-xx)} + R\,\text{arc.ſin.v.}\frac{x}{r}$. Facto $V = x\,\text{arc.ſin.v.}\frac{x}{r}$, fit arc. ſin. v. $\frac{x}{r} = \frac{dV}{dx} - \frac{x}{\sqrt{(2rx-xx)}}$;

$$\text{proinde } \frac{dS}{dx} = \sqrt{(2rx-xx)} + R\frac{dV}{dx} - R\frac{x}{\sqrt{(2rx-xx)}}$$

$$= \sqrt{(2rx-xx)} + R\frac{dV}{dx} + R\frac{r-x}{\sqrt{(2rx-xx)}} - \frac{Rr}{\sqrt{(2rx-xx)}};$$

$$\text{hinc } S = \tfrac{1}{2}rr\,\text{arc. ſin. v.}\frac{x}{r} - \tfrac{1}{2}(r-x)\sqrt{(2rx-xx)}$$

$$+ Rx\,\text{arc. ſin. v.}\frac{x}{r} + R\sqrt{(2rx-xx)}$$

$$- Rr\,\text{arc. ſin. v.}\frac{x}{r}$$

Sit $x = 2r$; erit $S = \frac{r(2R+r)}{2}\pi$.

 Sci-

Scilicet area cycloidis eſt ad aream circuli genitoris, ut $2R+r:r$; & nominatim cycloidis vulgaris area eſt tripla area circuli genitoris.

§. 103. Quoniam $\frac{dS}{dx}=y$; erit (§. 35.) juxta ſeriem Bernoullianam

$$S=xy-\frac{x^2}{1.2}\frac{dy}{dx}+\frac{x^3}{1.2.3}\frac{ddy}{dx^2}-\frac{x^4}{1.2\ldots4}\frac{d^3y}{dx^3}+\frac{x^5}{1.2\ldots5}\frac{d^4y}{dx^4}-\frac{x^6}{1.2\ldots6}\frac{d^5y}{dx^5}+\ldots$$

Cum autem quadratura curvarum una ſit ex præcipuis calculi integralis applicationibus, haud alienum ab re erit formulam hanc accuratius explicare, intimumque ejus cum methodo exhauſtionis nexum oſtendere.

Sit curva quæcunque ad axem aliquem relata, cujus curvæ ſegmentum coordinatis x & y terminatum denotet S. Abſciſſa axis in numerum quemcunque n partium Δx mutuo æqualium diviſa, inſcribantur & circumſcribantur curvæ rectangula ad normam §. 1. Ex. 3.

Rectæ axi ordinatim applicatæ, quæ abſciſſis x, $x-\Delta x$, $x-2\Delta x$, $x-3\Delta x$, $x-4\Delta x \ldots x-(n-1)\Delta x$ reſpondent, ſunt reſpective (§. 31.).

$$y$$

$$y-\frac{\Delta x}{1}\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2}-\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4}-\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5}+\ldots$$

$$y-2\frac{\Delta x}{1}\frac{dy}{dx}+2^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2}-2^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3}+2^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4}-2^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5}+\ldots$$

$$y-3\frac{\Delta x}{1}\frac{dy}{dx}+3^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2}-3^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3}+3^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4}-3^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5}+\ldots$$

$$y-4\frac{\Delta x}{1}\frac{dy}{dx}+4^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2}-4^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3}+4^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4}-4^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5}+\ldots$$

$$\vdots$$

$$y-(n-1)\frac{\Delta x}{1}\frac{dy}{dx}+(n-1)^2\frac{\Delta x^2}{1.2}\frac{ddy}{dx^2}-(n-1)^3\frac{\Delta x^3}{1.2.3}\frac{d^3y}{dx^3}+(n-1)^4\frac{\Delta x^4}{1.2..4}\frac{d^4y}{dx^4}-(n-1)^5\frac{\Delta x^5}{1.2..5}\frac{d^5y}{dx^5}+\ldots$$

Rectangula curvæ circumſcripta ſunt reſpective producta harum expreſſionum per altitudinem communem Δx; tandemque ſumma rectangulorum circumſcriptorum eſt ſumma omnium horum productorum, nempe

$y\Delta x.$

$$
\begin{array}{ll}
y\Delta x.n & = y\Delta x.n \\
- \frac{\Delta x^2}{1}\frac{dy}{dx}(1 + 2 + 3 + 4 \ldots n\text{-}1) & - \frac{\Delta x^2}{1}\frac{dy}{dx}(\text{ſ}.\, 1 \ldots n\text{-}1) \\
+ \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(1^2+2^2+3^2+4^2+\ldots(n\text{-}1)^2) & + \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(\text{ſ}.\, 1^2 \ldots n\text{-}1^2) \\
- \frac{\Delta x^4}{1.2.3}\frac{d^3y}{dx^3}(1^3+2^3+3^3+4^3+\ldots(n\text{-}1)^3) & - \frac{\Delta x^4}{1\ldots3}\frac{d^3y}{dx^3}(\text{ſ}.\, 1^3 \ldots n\text{-}1^3) \\
+ \frac{\Delta x^5}{1.2\ldots4}\frac{d^4y}{dx^4}(1^4+2^4+3^4+4^4+\ldots(n\text{-}1)^4) & + \frac{\Delta x^5}{1\ldots.4}\frac{d^4y}{dx^4}(\text{ſ}.\, 1^4 \ldots n\text{-}1^4) \\
- \frac{\Delta x^6}{1.2\ldots5}\frac{d^5y}{dx^5}(1^5+2^5+3^5+4^5+\ldots(n\text{-}1)^5) & - \frac{\Delta x^6}{1\ldots.5}\frac{d^5y}{dx^5}(\text{ſ}.\, 1^5 \ldots n\text{-}1^5). \\
+ \;\text{- - - - - - -} & + \;\text{- - - - -}
\end{array}
$$

$$
\begin{array}{l}
= y\Delta x.n \\
- \frac{\Delta x^2}{1}\frac{dy}{dx}(\tfrac{1}{2}nn + Bn) \\
+ \frac{\Delta x^3}{1.2}\frac{ddy}{dx^2}(\tfrac{1}{3}n^3 + Bnn + Cn) \\
- \frac{\Delta x^4}{1\ldots3}\frac{d^3y}{dx^3}(\tfrac{1}{4}n^4 + Bn^3 + Cn^2 + Dn) \\
+ \frac{\Delta x^5}{1\ldots4}\frac{d^4y}{dx^4}(\tfrac{1}{5}n^5 + Bn^4 + Cn^3 + Dn^2 + En) \\
- \frac{\Delta x^6}{1\ldots5}\frac{d^5y}{dx^5}(\tfrac{1}{6}n^6 + Bn^5 + Cn^4 + Dn^3 + En^2 + Fn) \\
+ \;\text{- - - - - - - -}
\end{array}
$$

$$
\begin{array}{l}
= xy \\
- \frac{1}{1}\frac{dy}{dx}(\tfrac{1}{2}x^2 + Bx\Delta x) \\
+ \frac{1}{1.2}\frac{ddy}{dx^2}(\tfrac{1}{3}x^3 + Bx^2\Delta x + Cx\Delta x^2) \\
- \frac{1}{1.2.3}\frac{d^3y}{dx^3}(\tfrac{1}{4}x^4 + Bx^3\Delta x + Cx^2\Delta x^2 + Dx\Delta x^3) \\
+ \frac{1}{1.2\ldots4}\frac{d^4y}{dx^4}(\tfrac{1}{5}x^5 + Bx^4\Delta x + Cx^3\Delta x^2 + Dx^2\Delta x^3 + Ex\Delta x^4) \\
- \frac{1}{1.2\ldots5}\frac{d^5y}{dx^5}(\tfrac{1}{6}x^6 + Bx^5\Delta x + Cx^4\Delta x^2 + Dx^3\Delta x^2 + Exx\Delta x^4 + Fx\Delta x^5) \\
+ \;\text{- - - - - - - - -}
\end{array}
$$

Proinde limitis ſummæ rectangulorum circumſcriptorum, hoc eſt, areæ ſegmenti curvæ expreſſio limes eſt ſummæ omnium horum terminorum; nempe

$$xy - \frac{x^2}{1.2}\frac{dy}{dx} + \frac{x^3}{1.2.3}\frac{ddy}{dx^2} - \frac{x^4}{1...4}\frac{d^3y}{dx^3} + \frac{x^5}{1...5}\frac{d^4y}{dx^4} - \frac{x^6}{1...6}\frac{d^5y}{dx^5} + \ldots.$$

Eademque methodo proceditur per rectangula curvæ inſcripta.

Scholium. Methodus, qua parallelogrammorum curvæ alicui circumſcriptorum ſumma determinata fuit, exempli loco eſſe poteſt methodi, qua ſerierum plurimarum ſumma, juvante theoremate Tayloriano, ad poteſtatum numerorum naturalium ſummas poteſt reduci.

§. 104. Si curva referatur ad aliquem focum, ut coordinatæ ſint radii vectores, & anguli, quos hi radii cum aliquo radio vectore poſitione dato comprehendunt: quadratura curvæ ſic determinatæ iisdem nititur principiis.

Fig. 26. Sit SMM' curva ad focum F relata per radios vectores FM, FM', & angulos SFM, SFM', quos radii vectores FM, FM' cum radio FS poſitione dato comprehendunt.

Centro F, radiis FM, FM', deſcribantur arcus Mm, $M'm'$, ut curvæ inſcribantur atque circumſcribantur ſectores circulares MFm, $M'Fm'$. Sector curvæ MFM' major eſt ſectore circulari inſcripto MFm; minor autem circumſcripto $M'Fm'$. Cum autem ratio æqualitatis limes ſit rationis radiorum FM, FM'; ratio æqualitatis limes etiam eſt rationis ſectorum circularium $M'Fm'$, MFm; & a fortiori limes rationis ſectoris curvæ MFM', & ſectoris circularis MFm. Radius magnitudine datus FS ſit r, & centro F radio FS deſcribatur arcus circularis, qui radiis FM & FM' in punctis X & X' occurrat. Sit etiam π circumferentia circuli, cujus radius eſt unitas; ſit angulus $SFM = x$; angulus $MFM' = \Delta x$, & ſectorem curvæ SFM denotet S; unde $MFM' = \Delta S$.

$$\begin{aligned}
&\text{lim.}MFM' : MFm &&= 1 : 1\\
&MFm : XFX' &&= MF^2 : FS^2\\
& &&= yy : rr\\
\text{ergo}\quad &\text{lim.}MFM' : XFX' &&= yy : rr\\
\text{ſeu}\quad &\text{lim.}MFM' : \tfrac{1}{2}rr\frac{\Delta x}{\pi} &&= yy : rr
\end{aligned}$$

lim.

$$\lim.\Delta S \; : \tfrac{1}{2} rr \frac{\Delta x}{\pi} = yy \; : \; rr$$

$$\lim.\frac{\Delta S}{\Delta x} \; : \tfrac{1}{2} rr \times \frac{1}{\pi} = yy \; : \; rr$$

ſeu $$\frac{dS}{dx} \; : \tfrac{1}{2} rr \times \frac{1}{\pi} = yy \; : \; rr$$

et $\frac{dS}{dx}$ - $= \frac{1}{2\pi} yy$, quæ eſt æquatio differentialis ſuperficierum curvarum ad focum per radios vectores relatarum.

Exempla. Sit $y = r\frac{x}{\pi}$, quæ eſt æquatio ſpiralis Archimedeæ.
$yy = rr\frac{xx}{\pi\pi}$, $\frac{1}{2\pi}yy = rr \times \frac{xx}{2\pi^3}$, $\frac{dS}{dx} = rr \times \frac{xx}{2\pi^3}$, $S = C + \tfrac{1}{3} rr \frac{x^3}{2\pi^3}$. Sit $S = 0$, quando $x = 0$; erit $S = \tfrac{1}{6} rr \frac{x^3}{\pi^3} = \tfrac{1}{6} yy \times \frac{x}{\pi} = \tfrac{1}{6}\frac{y^3}{r}$. Nempe ſuperficies ſpiralis Archimedeæ creſcit in ratione triplicata angulorum *SFM*, ſeu in ratione compoſita ex duplicata ratione radii vectoris *FM* & ratione ſimplici anguli *SFM*, ſeu tandem in ratione triplicata radii vectoris *FM*.

Sit $y = r\left(\frac{x}{\pi}\right)^m$; erit $yy = rr\left(\frac{x}{\pi}\right)^{2m}$, $\frac{dS}{dx} = \frac{1}{2\pi}yy = \frac{1}{2\pi}rr\left(\frac{x}{\pi}\right)^{2m}$,
$S = C + \frac{1}{2\pi}rr \cdot \frac{1}{2m+1}\frac{x^{2m+1}}{\pi^{2m}} = C + \tfrac{1}{2} \cdot \frac{1}{2m+1} rr \cdot \frac{x^{2m+1}}{\pi^{2m+1}} = C + \tfrac{1}{2}\frac{1}{2m+1} yy \cdot \frac{x}{\pi}$.
Superficies igitur omnium ſpiralium, quarum æquatio eſt $y = r\left(\frac{x}{\pi}\right)^m$, ſunt in ratione compoſita ex ratione angulorum *SFM* ſimplici, & ratione radiorum vectorum *FM* duplicata.

Sit $y = re^x$, quæ eſt æquatio ſpiralis logarithmicæ; erit $\frac{dS}{dx} = \tfrac{1}{2}yy = \tfrac{1}{2}rre^{2x}$; $S = \tfrac{1}{4}rre^{2x} = \tfrac{1}{4}yy$. Proinde ſuperficies ſpiralis logarithmicæ creſcit in ratione duplicata radiorum vectorum, ſeu etiam in progreſſione geometrica.

Sit $y = r\,\text{ſec.}^2\tfrac{1}{2}x$, quæ eſt æquatio parabolæ ad focum relatæ, in qua r æqualis quadranti parametri. Erit $\frac{dS}{dx} = \tfrac{1}{2}yy = \tfrac{1}{2}rr\,\text{ſec.}^4\tfrac{1}{2}x$; unde
$S = rr\,\text{tang.}\tfrac{1}{2}x\,(1 + \tfrac{1}{3}\text{tang.}^2\tfrac{1}{2}x) = rr\,(\text{tang.}\tfrac{1}{2}x + \tfrac{1}{3}\text{tang.}^3\tfrac{1}{2}x)$.

§. 105. Cum quadratura curvarum ad focum relatarum reducatur ad integrationem æquationis differentialis $\frac{dS}{dx} = yy$: quadratura hæc etiam poteſt obtineri per ſeriem Bernoullianam, prouti in §. 36. extenſa fuit.

Erit ſcilicet
$$2S = xyy$$
$$- 2\frac{x^2}{1.2}y\frac{dy}{dx}$$
$$+ 2\frac{x^3}{1.2.3}\left(\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right)$$
$$- 2\frac{x^4}{1.2...4}\left(3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right)$$
$$+ 2\frac{x^5}{1.2...5}\left(3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right)$$
$$- 2\frac{x^6}{1.2...6}\left(10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right)$$
$$+ \quad - \quad - \quad - \quad - \quad -$$

Si opus foret, eandem formulam ad methodum exhauſtionis immediate reducere, & ex primis limitum notionibus inferre liceret, uti in §. 103.

§. 106. Quemadmodum data curvæ æquatio ad ſuperficiei ipſius determinationem conducit; ita viciſſim cognitio hujus ſuperficiei poteſt relationem coordinatarum curvæ ſuppeditare. Quod paucis exemplis illuſtrare ſufficiat.

Sit $S = \frac{1}{m}xy$; erit $\frac{dS}{dx} = \frac{1}{m}y + \frac{1}{m}x\frac{dy}{dx}$

atqui $\frac{dS}{dx} = y$; ergo $y = \frac{1}{m}y + \frac{1}{m}x\frac{dy}{dx}$

et $my = y + x\frac{dy}{dx}$.

1°. Sit $m = 1$; fit $0 = x\frac{dy}{dx}$: unde $\frac{dy}{dx} = 0$, & $y = C$. Qui caſus eſt rectanguli, cujus altitudo magnitudine datur.

2°. Sit $m > 1$; fit $y(m-1) = x\frac{dy}{dx}$; ſeu $\frac{dx}{dy}\cdot\frac{1}{x} = \frac{1}{m-1}\cdot\frac{1}{y}$

unde $C + \log. x = \frac{1}{m-1}\log. y$.

$\log. ax = \frac{1}{m-1}\log. y$. $y = a^{m-1}x^{m-1}$.

Si

Si $m = 2$; fit $y = ax$, quæ eſt æquatio ad lineam rectam: in reliquis caſibus curva eſt parabolica, ad axem aut ad tangentem ex vertice ductam relata.

3°. Sit $m < 1$; fit $y(1-m) + x\frac{dy}{dx} = 0$; unde $(1-m)\log.y + \log.x = \log.C$; $y^{1-m}x = C$, quæ eſt æquatio hyperbolarum ad aſymptoton relatarum.

Sit $S = t(b-y)$, $\frac{dS}{dx} = -t\frac{dy}{dx}$. Sed $\frac{dS}{dx} = y$: ergo $y = -t\frac{dy}{dx}$, & $1 = -\frac{t}{y} \times \frac{dy}{dx}$; hinc $x = C - t\log.y$. Sit $x = a$, quando $y = b$; $x = t\log.b - t\log.y$, ſeu $\frac{x}{t} = \log.\frac{b}{y}$, quæ eſt æquatio curvæ logarithmicæ.

Curva referatur ad aliquem focum; & ſit $S = yy \times \frac{x}{r}$. Erit $\frac{dS}{dx} = yy \times \frac{1}{r} + 2y\frac{dy}{dx} \times \frac{x}{r}$. Atqui $\frac{dS}{dx} = yy$ (§. 104): ergo $yy(1 - \frac{1}{r}) = 2y\frac{dy}{dx} \times \frac{x}{r}$; ſeu $y(r-1) = 2x\frac{dy}{dx}$, & $\frac{dx}{dy} \times \frac{1}{2x} = \frac{1}{y} \times \frac{1}{r-1}$; hinc $C + \frac{1}{2}\log.x = \frac{1}{r-1}\log.y$; ſeu $\frac{1}{2}\log.ax = \frac{1}{r-1}\log.y$, $ax = y^{\frac{2}{r-1}}$, quæ eſt æquatio ad ſpirales vulgares.

Sit $S = \frac{p}{q}yy$; $\frac{dS}{dx} = 2\frac{p}{q}y\frac{dy}{dx} = yy$; $2\frac{p}{q}.\frac{dy}{dx} = y$; $\frac{1}{2}\frac{q}{p}.\frac{dx}{dy} = \frac{1}{y}$; $C + \frac{1}{2}\frac{q}{p}x = \log.y$, quæ eſt æquatio ad ſpiralem logarithmicam.

Appendix.

De quadratura hyperbolæ conicæ.

Ne ſeries propoſitionum principalium, hoc capite contentarum, abrumperetur; ſatius eſſe ratus ſum particularem hyperbolæ conicæ quadraturam geometrice conſideratam ſeorſim exponere, ut ſequitur. (Vid. §. 99.)

§. 107. *Lemma.* Si curva quæcunque diametro ſit prædita (hoc eſt, ſi datur recta, quæ omnes rectas ſibi mutuo parallelas & curva utrinque terminatas bifariam ſecat): dico, hanc diametrum ſpatium quoque curvilineum in duas partes æquales dividere.

Etenim curvæ inſcribantur & circumſcribantur parallelogramma (ad normam §. 1. Ex. 3.). Parallelogramma hæc bina ſumpta ad utramque diametri partem ſunt mutuo æqualia; proinde & ſummæ horum parallelogrammorum ad utramque diametri partem ſunt invicem æquales: unde & limites harum ſummarum, nempe duæ partes ſpatii curvilinei ad utramque diametri partem ſitæ, æquales ſunt.

Fig. 27. *Theorema.* Super aſymptoto hyperbolæ conicæ ſumantur a centro C rectæ CA, CA', CB, CB' in proportione geometrica. Per puncta A, A', B, B' agantur rectæ alteri aſymptoto parallelæ, quæ hyperbolæ in punctis D, D', E, E' occurrant. Ex centro C agantur rectæ CD, CD', CE, CE'; dico, ſectores DCD', ECE', & trapezia $ADD'A'$, $BEE'B'$, eſſe invicem æqualia.

Jungantur rectæ ED', $E'D$, quæ aſymptotis in G, H, G', H' reſpective occurrant. Notum eſt, fore $GE = D'H$, $G'E' = DH'$.

Propter parallelas EB, CH eſt $GB : GE = A'C : D'H$

atqui $GE = D'H$

ergo $GB = A'C$; & pariter $B'G' = AC$.

Atqui (hyp.) $CA : CA' = CB : CB' = E'B' : EB$ (quoniam $CB \times BE = CB' \times B'E'$);

ergo $G'B' : GB = E'B' : EB$; proinde triangula $G'B'E'$, GBE ſunt invicem ſimilia, & rectæ $G'E'$, GE ſunt invicem parallelæ. Ducatur CF, quæ rectam GH bifariam in F ſecet. Quoniam $GF : G'F' = CF : CF' = FH : F'H'$; erit etiam $G'F' = F'H'$. Atqui $GE = D'H$, & $G'E' = DH'$; ergo $EF = FD'$, & $E'F' = F'D$. Proinde recta CF eſt diameter hyperbolæ, & ſpatia SEF, $SE'F'$ reſpective æqualia ſunt ſpatiis $SD'F$, SDF'. Sed triangula CEF, $CE'F'$ reſpective æqualia ſunt triangulis $CD'F$, CDF'; ergo & ſpatia CES, $CE'S$ reſpective æqualia ſunt ſpatiis $CD'S$, CDS; proinde & ſectores ECE', DCD' ſunt invicem æquales.

Rectæ AD, CD' ſibi invicem occurrant in I. Quoniam $CA \times AD = CA' \times A'D'$ ſemper eſt triangulum CAD æquale triangulo $CA'D'$: hinc, ſublato utrinque ſpatio communi CAI, & adjecto utrinque ſpatio communi DID', fit ſector DCD' æqualis trapezio mixtilineo $ADD'A'$; eodemque modo ſector ECE' æqualis eſt trapezio $BEE'B'$.

Corol.

Corollarium primum. Super aſymptoto CA ſumantur abſciſſæ quotcunque in progreſſione geometrica, & ducantur ordinatæ — Fig. 28.

$CA,\ CA',\ CA'',\ CA''',\ CA^{\text{iv}} \ldots CA^{N}$

$AB,\ A'B',\ A''B'',\ A'''B''',\ A^{\text{iv}}B^{\text{iv}} \ldots A^{N}B^{N}$

Omnia trapezia $AB',\ A'B'',\ A''B''',\ A'''B^{\text{iv}},\ A^{\text{iv}}B^{\text{v}} \ldots A^{N-1}B^{N}$

ſunt invicem æqualia; & proinde ſpatia hyperbolica AB^{N} creſcunt ut logarithmi abſciſſarum CA^{N}, CA.

Corollarium ſecundum. Datis in aſymptoto duobus punctis quibuscunque A, A^{N}; nullus eſt limes numeri abſciſſarum in progreſſione geometrica, cujus exponens $CA^{N} : CA$, creſcentium, quæ in aſymptoto producta CA ſumi poſſunt. Quare nullus etiam eſt limes numeri ſpatiorum trapezio AB^{N} æqualium, quæ in ſpatio aſymptotico ſumi poſſunt; ideoque nullus eſt limes magnitudinis hujus ſpatii.

Corollarium tertium. Detur ratio $CA^{N} : CA$. Ratio hæc dividatur in quotcunque n rationes $CA' : CA$ ſibi invicem æquales; proindeque trapezium hyperbolicum AB^{N} dividatur in eundem numerum partium æqualium AB', $A'B''$, $A''B''' \ldots A^{N-1}B^{N}$. Ducantur Bb', $B'b$ aſymptoto CA parallelæ, quæ ipſis $A'B'$, $A''B''$ in b' & b occurrant. Quoniam

$$CA : CA' = CA' : CA''$$

$$CA'-CA : CA''-CA' = CA : CA'$$

ſeu $AA' : A'A'' = A'B' : AB$, & $AA' \times AB = A'A'' \times A'B'$; & ſic deinceps: unde omnia parallelogramma ſpatio AB^{N} circumſcripta ſunt invicem æqualia. Quoniam ratio $CA^{N} : CA$ datur: eodem manente numero n, pariter datur ratio $CA' : CA$; adeoque & ratio $AA' : CA$, ſeu ratio parallelogrammi $ABb'A'$ ad potentiam hyperbolæ $CA \times AB$ ſin. A. Et proinde etiam datur ratio ſummæ omnium parallelogrammorum ſpatio AB^{N} circumſcriptorum ad potentiam hyperbolæ; & proinde limes ſummæ horum parallelogrammorum ſeu ſpatium AB^{N} eſt etiam ad potentiam hyperbolæ in eadem ratione data. In diverſis itaque hyperbolis trapezia, duabus ordinatis, aut duabus abſciſſis, quarum ratio datur, reſpondentia, ſunt inter ſe uti potentiæ harum hyperbolarum; ſeu potentia hyperbolæ eſt modulus ſyſtematis logarithmici ad hanc hyperbolam pertinentis.

Applicatio 1. Hinc conſequitur menſura ſpatii hyperbolici inter duas rectas aſymptoto ordinatim applicatas, quarum ratio datur, comprehenſi. Scilicet ſumatur logarithmus naturalis rationis harum ordinatarum, & per hunc logarithmum multiplicetur potentia hyperbolæ.

Fig. 27. *Applicatio* 2. Hinc etiam fluit menſura ſpatii cujuslibet hyperbolici. Sit v. gr. $CS = a$ ſemi-axis transverſus hyperbolæ, & quæratur ſpatium ESF. Sit $Sb = b$ ſemi-axis ſecundus; & ſit Sb' aſymptoto CB ordinatim applicata, ſeu ipſi EB parallela; atque potentia hyperbolæ, nempe $Cb' \times b'S\,\text{ſin}.C$, ſeu $\frac{1}{2}ab$, dicatur P.

$$ESF = ECF - ECS = ECF - EBb'S = ECF - P\log.\frac{Sb'}{EB} = ECF + P\log.\frac{EB}{Sb'}$$
$$= ECF + P\log.\frac{GE}{Sb} = ECF + P\log.\frac{GF - EF}{Sb} = ECF + P\log.\left(\frac{CF}{CS} - \frac{EF}{Sb}\right)$$
$$= \tfrac{1}{2}xy + P\log.\left(\frac{x}{a} - \frac{y}{b}\right) = \tfrac{1}{2}xy + P\log.\frac{bx - ay}{ab} = \tfrac{1}{2}\frac{b}{a}x\sqrt{(xx - aa)} + \tfrac{1}{2}ab\log.\frac{x - \sqrt{(xx - aa)}}{a}$$
$$= \tfrac{1}{2}\frac{b}{a}x\sqrt{(xx - aa)} + \tfrac{1}{2}ab\log.\frac{a}{x + \sqrt{(xx - aa)}}.$$

Caput Undecimum.

De rectificatione curvarum.

§. 108.

Ratio æqualitatis limes eſt rationis arcus cujusvis curvæ ad chordam ſuam (§. 47.). Hoc principio nititur rectificatio curvarum.

Etenim ſint x & y abſciſſa axis & recta axi huic ordinatim applicata ſub angulo recto. Sit S arcus curvæ: eſt $\lim.\Delta S : \sqrt{(\Delta x^2 + \Delta y^2)} = 1 : 1$

Fig. 11.

$$\text{ſeu} \quad \lim.\frac{\Delta S}{\Delta x} : \sqrt{\left(1 + \frac{\Delta y^2}{\Delta x^2}\right)} = 1 : 1$$

$$\text{ſeu} \quad \frac{dS}{dx} : \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = 1 : 1$$

$$\text{\& proinde} \quad \frac{dS}{dx} = \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}.$$

Eadem

Eadem æquatio differentialis poteſt etiam ex §. 49. obtineri. Quoniam eſt $\frac{dx}{dS} = \text{ſin.}PMT$, eſt $\frac{dS}{dx} = \text{coſec.}PMT$: ſed $\text{cot.}PMT = \frac{dy}{dx}$ (§. 49.); proinde $\text{coſec.}PMT = \sqrt{(1 + (\frac{dy}{dx})^2)}$; hinc $\frac{dS}{dx} = \sqrt{(1 + (\frac{dy}{dx})^2)}$.

Quodſi autem angulus coordinatarum non eſt rectus, ponatur hic angulus $= \varphi$; eritque eodem modo $\frac{dS}{dx} = \sqrt{(1 - 2\frac{dy}{dx}\text{coſ.}\varphi + (\frac{dy}{dx})^2)}$.

Data igitur æquatione curvæ rectificatio ejus ad calculum integralem reducitur.

Exemplum 1. Sit $pyy = x^3$, ſeu $p^{\frac{1}{2}}y = x^{\frac{3}{2}}$; erit $\frac{dS}{dx} = \frac{1}{2}\sqrt{\frac{4p+9x}{p}}$; unde $S = C + \frac{1}{27}\frac{(4p+9x)^{\frac{3}{2}}}{\sqrt{p}}$. Sit $S = 0$, quando $x = 0$; $S = \frac{1}{27}\left(\frac{(4p+9x)^{\frac{3}{2}}}{\sqrt{p}} - 8p\right)$. Proinde parabolæ, æquatione $pyy = x^3$ determinatæ, rectificatio abſolute habetur.

Exemplum 2. Sit $yy = aa - xx$, ſeu $y = \sqrt{(aa - xx)}$: hinc $\frac{dS}{dx} = \frac{a}{\sqrt{(aa-xx)}}$, quæ eſt æquatio differentialis arcus circuli. Fit itaque

$$\frac{dS}{dx} = 1 + \frac{1}{2}\cdot\frac{xx}{aa} + \frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{x^4}{a^4} + \frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{x^6}{a^6} + \frac{1}{2^4}\cdot\frac{1...7}{1...4}\cdot\frac{x^8}{a^8} + \ldots$$

$$S = x + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{x^3}{aa} + \frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{1}{5}\cdot\frac{x^5}{a^4} + \frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{1}{7}\cdot\frac{x^7}{a^6} + \frac{1}{2^4}\cdot\frac{1...7}{1...4}\cdot\frac{1}{9}\cdot\frac{x^9}{a^8} + \ldots$$

Sit $x = a = 1$; $p = 1 + \frac{1}{2}\cdot\frac{1}{3}\cdot + \frac{1}{2^2}\cdot\frac{1.3}{1.2}\cdot\frac{1}{5} + \frac{1}{2^3}\cdot\frac{1.3.5}{1.2.3}\cdot\frac{1}{7} + \frac{1}{2^4}\cdot\frac{1...7}{1...4}\cdot\frac{1}{9} + \ldots$

Exemplum 3. Sit $yy = 4px$, $y = 2\sqrt{px}$, $\frac{dy}{dx} = \sqrt{\frac{p}{x}}$; $\frac{dS}{dx} = \sqrt{1 + \frac{p}{x}}$, quæ eſt æquatio differentialis arcus parabolæ. Unde erit

$$S = \frac{y}{2p}\sqrt{(4pp+yy)} + 2p\,\text{log.}\frac{2p}{\sqrt{(4pp+yy)} - y}.$$

Exemplum 4. Sit $yy = \frac{bb}{aa}(aa - xx)$, $\frac{dS}{dx} = \frac{e}{a}\cdot\frac{\sqrt{(\frac{a^4}{ee} - xx)}}{\sqrt{(aa-xx)}}$, poſito $ee = a^2 - b^2$, quæ eſt æquatio differentialis arcus ellipſeos. Hanc formulam terminis numero finitis integrare, aut ſaltem ad arcus circulares aut ad logarithmos terminis finitis reducere, fruſtra huc usque fuit tentatum.

 Nume-

Numerator $\mathcal{V}\left(\frac{a^4}{ee}-xx\right)=a\mathcal{V}\left(\frac{aa}{ee}-\frac{xx}{aa}\right)$ in ſeriem convertatur; fit

$$\mathcal{V}\left(\frac{a^4}{ee}-xx\right)=a\left(\frac{a}{e}-\frac{1}{2}\cdot\frac{e}{a}\cdot\frac{xx}{aa}-\frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\left(\frac{e}{a}\right)^3\cdot\left(\frac{x}{a}\right)^4-\frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\left(\frac{e}{a}\right)^5\cdot\left(\frac{x}{a}\right)^6-\ldots\right.$$

unde $\frac{dS}{dx}=\frac{a}{\mathcal{V}\,aa-xx}\left(1-\frac{1}{2}\cdot\frac{ee}{aa}\cdot\frac{xx}{aa}-\frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\left(\frac{e}{a}\right)^4\cdot\left(\frac{x}{a}\right)^4-\frac{1}{2^4}\cdot\frac{1.3}{1.2.3}\cdot\left(\frac{e}{a}\right)^6\cdot\left(\frac{x}{a}\right)^6\ldots\right.$

Integratio uniuscujusque termini hujus ſeriei reducitur ad arcus circulares; & nominatim, facto $x=a$, quadrans perimetri ellipſeos reducitur ad quadrantem circumferentiæ circuli ſerie regulari, quæ ſequitur.

$$S=ap\left(1-\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{ee}{aa}-\frac{1}{2^2}\cdot\frac{1}{1.2}\cdot\frac{3.1}{4.2}\cdot\frac{e^4}{a^4}-\frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\frac{5.3.1}{6.4.2}\cdot\frac{e^6}{a^6}-\frac{1}{2^4}\cdot\frac{1.3.5}{1\ldots4}\cdot\frac{7\ldots1}{8\ldots2}\cdot\frac{e^8}{a^8}-\ldots\right.$$

Quæ ſeries promte convergit, ſi exigua fuerit ellipſeos excentricitas e.

Æquatio differentialis rectificationis ellipſeos poteſt etiam paulo ſimplicius obtineri, ut ſequitur. Sint CA, CB dimidii axes ellipſeos. Centro C radio CA (v. gr.) deſcribatur quadrans circumferentiæ AND, cui CB occurrat in D. Tum angulo ACN dicto z, erit $CP=a\,\text{ſin.}z$, $NP=a\,\text{coſ.}z$, $MP=b\,\text{coſ.}z$, Fig. 29.

$\frac{d.CP}{dz}=a\,\text{coſ.}z$, $\frac{d.MP}{dz}=-b\,\text{ſin.}z$; hinc

$$\frac{dS}{dz}=\mathcal{V}(aa\,\text{coſ.}^2z+bb\,\text{ſin.}^2z)=a\mathcal{V}\left(1-\frac{ee}{aa}\text{ſin.}^2z\right)=b\mathcal{V}\left(1+\frac{ee}{bb}\text{coſ.}^2z\right)$$

$$=a\left(1-\frac{1}{2}\frac{ee}{aa}\text{ſin.}^2z-\frac{1}{2^2}\cdot\frac{1}{2}\frac{e^4}{a^4}\text{ſin}^4z-\frac{1}{2^3}\cdot\frac{1.3}{1.2.3}\cdot\frac{e^6}{a^6}\text{ſin.}^6z-\frac{1}{2^4}\cdot\frac{1.3.5}{1\ldots4}\cdot\frac{e^8}{a^8}\text{ſin.}^8z-\ldots\right.$$

Facto $z=90$ oritur eadem ſeries, quam prius obtinuimus.

Exemplum 5. Sit $yy=\frac{bb}{aa}(xx-aa)$, fit $\frac{dS}{dx}=\frac{e}{a}\cdot\frac{\mathcal{V}\,xx-\frac{a^4}{ee}}{\mathcal{V}\,xx-aa}$, poſito $ee=bb+aa$, quæ eſt æquatio differentialis arcus hyperbolæ: numeratore in ſeriem converſo integratio ad logarithmos reducitur.

Exemplum 6. Sit $y=\mathcal{V}(2rx-xx)+r\,\text{arc.ſin.v.}\frac{x}{r}$, quæ eſt æquatio cycloidis vulgaris. $\frac{dy}{dx}=\frac{r-x}{\mathcal{V}(2rx-xx)}+\frac{r}{\mathcal{V}(2rx-xx)}=\frac{2r-x}{\mathcal{V}(2rx-xx)}=\mathcal{V}\frac{2r-x}{x}$,

$\frac{dS}{dx}=\mathcal{V}\frac{2r}{x}$, $S=2\mathcal{V}rx$. Facto $x=2r$, $S=4r$.

Fit

Exemplum 7. Sit $y = \sqrt{(2rx - xx)} + R\,\text{arc.ſin.v.}\frac{x}{r}$, quæ eſt æquatio cycloidum protractarum aut contractarum.

Fit $\frac{dS}{dx} = \frac{\sqrt{(RR+rr+2R(r-x)}}{\sqrt{(rr-(r-x)^2)}}$. Hæc formula reducitur ad rectificationem ellipſeos methodo ſequenti.

Fiat $r - x = \frac{AA - zz}{2B}$; erit $\frac{dS}{dz} = \frac{z}{B} \cdot \frac{\sqrt{(RR+rr+\frac{AA}{B}R - \frac{R}{B}zz)}}{\sqrt{\left(\left(r+\frac{AA-zz}{2B}\right)\left(r-\frac{AA-zz}{2B}\right)\right)}}$:

fiat $AA = 2Br$; erit $\frac{dS}{dz} = 2\frac{\sqrt{((R+r)^2 - \frac{R}{B}zz)}}{\sqrt{(4Br - zz)}}$:

fiat $B = R$; erit $\frac{dS}{dz} = 2\frac{\sqrt{((R+r)^2 - zz)}}{\sqrt{(4Rr - zz)}}$. Unde res ad rectificationem ellipſeos reducitur. (Ex. 4.)

§. 109. Rectificatio curvarum ad focum aliquem relatarum eodem modo determinatur, aut etiam paulo aliter, ut ſequitur.

Sin. $FMT = y\frac{dx}{dS}$ (§. 49.); hinc $\frac{dS}{dx} = y\,\text{coſec.}FMT$. Atqui Fig. 15.

$\text{cot.}FMT = \frac{1}{y} \cdot \frac{dy}{dx}$; & proinde $\text{coſec.}FMT = \sqrt{\left(1 + \frac{1}{yy} \cdot \left(\frac{dy}{dx}\right)^2\right)}$; hinc

$\frac{dS}{dx} = \sqrt{\left(yy + \left(\frac{dy}{dx}\right)^2\right)}$.

Exemplum 1. Sit $y = r \cdot \left(\frac{b}{a}\right)^x$, quæ eſt æquatio ſpiralis logarithmicæ.

$\frac{dy}{dx} = r\log.\frac{b}{a} \cdot \left(\frac{b}{a}\right)^x$, $\frac{dS}{dx} = r\left(\frac{b}{a}\right)^x \sqrt{\left(1 + \log.^2\frac{b}{a}\right)}$; $S = y\sqrt{\left(1 + \log.^2\frac{b}{a}\right)}$: unde etiam arcus ſpiralis logarithmicæ creſcunt in progreſſione geometrica, dum anguli creſcunt in progreſſione arithmetica.

Exemplum 2. Sit $y = r\,\text{ſec.}^2\tfrac{1}{2}x$, quæ eſt æquatio parabolæ conicæ. Erit $\frac{dS}{dx} = r\,\text{ſec.}^3\tfrac{1}{2}x$. $S = r(\text{ſec.}\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x + \log.(\text{ſec.}\tfrac{1}{2}x + \text{tang.}\tfrac{1}{2}x)) = r(\text{ſec.}\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x + \log.\,\text{tang.}45° + \tfrac{1}{2}x)$.

Exemplum 3. Sit $y = \frac{aa - ee}{a + e\,\text{coſ.}x}$, quæ eſt æquatio focalis ellipſeos conicæ;

$\frac{dS}{dx} = \frac{aa - ee}{(a + e\,\text{coſ.}x)^2}\sqrt{(aa + 2ae\,\text{coſ.}x + ee)} = \frac{aa - ee}{(a + e\,\text{coſ.}x)^2}\sqrt{((aa+ee)^2 - 4ae\,\text{ſin.}^2x)}$.

§. 110.

§. 110. Quoniam $\frac{dS}{dx} = \sqrt{(1+(\frac{dy}{dx})^2)}$; rectificatio cujusvis curvæ exprimi etiam poteſt per ſeriem Bernoullianam. Plerumque autem exponentes differentiales altiorum ordinum, quibus hæc ſeries conſtaret, adeo fiunt complexi, ut reductionis hujus utilitas valde dubia fiat.

Majoris momenti eſt obſervatio, quæ curvæ cujusvis rectificationem ad quadraturam alterius curvæ reducere docet.

Fig. 30. Sit nempe altera curva $M'm'$ ſuper eodem axe deſcripta, cujus ſuperficies ſit S', & ordinatæ y'. Erit (§. 100.) $\frac{dS'}{dx} = y'$, $\frac{dS}{dx} = \sqrt{(1+(\frac{dy}{dx})^2)}$. Ergo ſi fiat ſemper $y' = a\sqrt{(1+(\frac{dy}{dx})^2)}$; erit $\frac{dS'}{dx} = a.\frac{dS}{dx}$; & proinde $S' = aS$, ſi S & S' evaneſcant ſimul facto $x = 0$.

Natura autem alterius hujus curvæ per priorem facilius determinatur modo ſequente.

Definitio. Sit curva ad aliquem axem relata per rectas ipſi perpendiculariter ordinatim applicatas. Ad punctum aliquod hujus curvæ ducatur recta ipſam tangens. Tum per idem punctum ducatur recta tangenti perpendicularis; quæ axi (ſi fieri poſſit) occurrat. Pars hujus perpendicularis, punctum inter contactus & axem intercepta, dicitur *normalis;* & pars axis, normalem inter & rectam axi ex eodem puncto ordinatim applicatam comprehenſa, dicitur *ſubnormalis.*

Sit nempe MN tangenti MT perpendicularis, & quæ axi occurrat in N; linea MN dicitur *normalis*, & PN dicitur *ſubnormalis.*

In triangulo rectangulo TMN eſt $TP : PM = PM : PN$; proinde $PN = \frac{yy}{y\frac{dx}{dy}} = y\frac{dy}{dx}$. Hinc $TN = y(\frac{dx}{dy} + \frac{dy}{dx})$. Atqui $MN^2 = TN \times PN$; ergo $MN^2 = yy(1+(\frac{dy}{dx})^2)$, & $MN = y\sqrt{(1+(\frac{dy}{dx})^2)}$.

Quoniam autem $M'P = y' = a\sqrt{(1+(\frac{dy}{dx})^2)}$; fit $y' = a \times \frac{MN}{y}$, ſeu $y : MN = a : y'$. Proinde poſterioris curvæ ordinata eſt quarta proportionalis ordinatæ & normali prioris curvæ, & alicui rectæ magnitudine datæ.

Exem-

Exempla. 1°. Sit $yy=2px$; $MN=y\sqrt{(1+\frac{pp}{yy})}=\sqrt{(pp+yy)}$; $y'=a.\frac{\sqrt{pp+yy}}{y}$.
$a:y'=y:\sqrt{pp+yy}$; $aa:y'y'=yy:pp+yy$.

$$aa:y'y'-aa=yy:pp=2px:pp=2x:p;\ y'y'=aa(1+\frac{p}{2x}).$$

2°. Sit $ayy=x^3$, $\frac{dy}{dx}=\frac{3xx}{2ay}$: $y'y'=\frac{1}{4}a(4a+9x)=\frac{9}{4}a(\frac{4}{9}a+x)$. Quoniam poſterior curva, nempe parabola conica, eſt quadrabilis; prior curva eſt rectificabilis. (Vide §. 108. Ex. 1.)

§. 111. Data expreſſione arcus alicujus curvæ per coordinatas, ejus determinari poteſt ipſa curva (quantum permittit imperfecta calculi integralis conditio).

Exemplum 1. Quæratur linea, cujus longitudo eſt abſciſſæ proportionalis. Quoniam eſt $S=\frac{m}{n}x$, $\frac{dS}{dx}=\frac{m}{n}$; atqui $\frac{dS}{dx}=\sqrt{(1+(\frac{dy}{dx})^2)}$; ergo
$1+(\frac{dy}{dx})^2=\frac{mm}{nn}$, $(\frac{dy}{dx})^2=\frac{mm-nn}{nn}$, $\frac{dy}{dx}=\frac{\sqrt{(mm-nn)}}{n}$; $y=C+x\frac{\sqrt{(mm-nn)}}{n}$, quæ eſt æquatio ad lineam rectam.

Exemplum 2. Quæritur curva, cujus arcus creſcit in ratione ſubduplicata abſciſſæ. Eſt ideo $S=2\sqrt{ax}$, $\frac{dS}{dx}=\sqrt{\frac{a}{x}}=\sqrt{(1+(\frac{dy}{dx})^2)}$; hinc
$\frac{a}{x}=1+(\frac{dy}{dx})^2$, $\frac{dy}{dx}=\sqrt{\frac{a-x}{x}}=\frac{a-x}{\sqrt{ax-xx}}=\frac{\frac{1}{2}a}{\sqrt{ax-xx}}-\frac{\frac{1}{2}a-x}{\sqrt{ax-xx}}$,
$y=\frac{1}{2}a\times\text{arc. ſin.v.}\frac{x}{\frac{1}{2}a}+\sqrt{(ax-xx)}$, quæ eſt æquatio cycloidis vulgaris.

CAPUT DUODECIMUM.

De capacitate ſolidorum rotundorum.

§. 112.

Sit curva quælibet ad axem aliquem relata per rectas axi huic perpendiculariter (v. gr. *a*) applicatas. Sit etiam rectangulum datam habens baſim *b*, & cujus

a) Quæcunque dicentur de caſu, quo rectæ axi ordinatim applicantur ad angulos rectos, facillime applicantur caſui, quo eædem ſub dato angulo obliquo axi occurrunt.

cujus altitudo mutabilis ſit abſciſſæ axis æqualis. Rectangulum & ſpatium curvilineum eidem aſciſſæ reſpondentia circa hujus axem ſimul revolvantur. Dico: limitem rationis, quæ inter mutationes ſimultaneas cylindri a rectangulo, & ſolidi a ſpatio curvilineo rotatione hac ſimul genitorum intercedit, æqualem eſſe rationi duplicatæ baſis rectanguli & ordinatæ huic abſciſſæ reſpondentis.

Fig. 24. Sit SMP curva ad axem SP per rectas MP huic perpendiculariter applicatas relata. Sit SA recta magnitudine data eidem axi perpendiculariter applicata. Sit SP abſciſſa axis, cui reſpondet ordinata MP; & compleatur rectangulum $SARP$, cujus altitudo creſcit ut abſciſſa SP. Spatium curvilineum SMP & rectangulum $SARP$ circa axem SP ſimul revolvantur. Dico, limitem rationis mutationum ſimultanearum cylindri a rectangulo $SARP$ & ſolidi a curva SMP ſimul genitorum æqualem eſſe rationi $SA^2 : MP^2$.

Sit PP' mutatio abſciſſæ axis; & proinde mutationes ſimultaneæ cylindri & prædicti ſolidi ſint ſolida rotatione rectanguli $SRR'P'$ & ſpatii $PMM'P'$ ſimul genita. Curvæ SMP inſcribantur & circumſcribantur rectangula $PMmP'$, $Pm'M'P'$. Solidum a ſpatio $PMM'P'$ genitum minus eſt cylindro a rectangulo $Pm'M'P'$ genito; majus autem cylindro ſimul genito a rectangulo inſcripto $PMmP'$. Atqui hi duo cylindri æque alti ſunt in ratione duplicata ordinatarum $M'P'$, MP: & proinde imminuta altitudine PP' ratio æqualitatis limes eſt rationis horum cylindrorum (§. 14.); tantoque magis ratio æqualitatis limes eſt rationis alterutrius horum cylindrorum ad ſolidum rotatione ſpatii $MPP'M'$ genitum. Quare ſolido a curva SMP genito dicto S, fit

$$\begin{array}{lllcl}
 & \text{lim.}\,\Delta S & : \text{cyl.}\,PMmP' & = & 1 : 1 \\
\text{Atqui} & \text{cyl.}PMmP' & : \text{cyl.}PRR'P' & = & PM^2 : SA^2 \\
\text{Quare} & \text{lim.}\,\Delta S & : \text{cyl.}PRR'P' & = & PM^2 : SA^2 \ (\text{§. } 14.) \\
\text{ſeu} & \text{lim.}\,\Delta S & : \pi . SA^2 . PP' & = & PM^2 : SA^2 \\
 & & & = & \pi . PP' \times PM^2 : \pi . PP' . SA^2 ; \\
\text{unde} & \text{lim.}\,\Delta S & & = & \pi . PP' \times PM^2 \\
\text{et} & \text{lim.}\,\frac{\Delta S}{dx} & & = & \pi . PM^2 \\
\text{ſeu} & \frac{dS}{dx} & & = & \pi . PM^2 = \pi . yy .
\end{array}$$

Obſer-

Obſervatio. Eadem expreſſio differentialis eodem modo obtinetur, ſi ſpatium curvilineum $ABPM$, duabus ordinatis AB, MP, abſciſſa axis BP, & arcu AM terminatum, circa axem SP revolvatur. Fig. 25.

Exemplum 1. Sit $y = \frac{x^m}{a^{m-1}}$; erit $yy = \frac{x^{2m}}{a^{2m-2}} = \frac{dS}{dx}$, $S = C + \frac{1}{2m+1}\frac{x^{2m+1}}{a^{2m-2}}$.

1°. Sit m numerus poſitivus; & ſit $S = 0$, quando $x = 0$: erit $S = \frac{1}{2m+1}\frac{x^{2m+1}}{a^{2m-1}} = \frac{1}{2m+1}xyy$. Proinde data parabola, cujus æquatio eſt $y = \frac{x^m}{a^{m-1}}$; paraboloides dimidio parabolæ ſegmento circa axem revoluto genita eſt ad cylindrum æque altum ſuper eadem baſi, in ratione data $1 : 2m+1$. Fig. 24.

Scholium. Exemplo hoc continetur caſus, quo ſpatium revolutum eſt triangulum, facto nempe $m = 1$.

2°. Sit m numerus negativus, ſeu ſit $y(a+x)^m = ba^m$. Ergo $\frac{dS}{dx} = bba^{2m}(a+x)^{-2m}$; Fig. 25.

$$S = C + \frac{1}{1-2m}bba^{2m}(a+x)^{-2m+1} = C + \frac{1}{1-2m}\cdot\frac{bba^{2m}}{(a+x)^{2m-1}} = C + \frac{1}{1-2m}yy(a+x).$$

Sit $S = 0$, quando $x = 0$, $y = b$; erit $S = \frac{1}{1-2m}(yy(a+x) - abb) = \frac{1}{2m-1}(abb - yy(a+x))$.

1°. Sit $2m > 1$, ſeu $m > \frac{1}{2}$. Tum $S = \frac{1}{2m-1}(abb - yy(a+x))$, ſeu ſolidum rotatione ſpatii hyperbolici $ABPM$ genitum proportionale eſt differentiæ cylindrorum a rectangulis $ABSD$, $MPSQ$ genitorum. Quoniam autem $y = \frac{ba^m}{(a+x)^m}$, $yy = \frac{bba^{2m}}{(a+x)^{2m}}$, & $yy(a+x) = \frac{bba^{2m}}{(a+x)^{2m-1}}$; nec ullus eſt limes magnitudinis abſciſſæ $a+x$: nullus eſt limes parvitatis quantitatis $\frac{bba^{2m}}{(a+x)^{2m-1}}$, ſeu cylindri $yy(a+x)$. Hinc in formula $S = \frac{1}{2m-1}(abb - yy(a+x))$ ſolidum S habet limitem magnitudinis, nempe $\lim. S = \frac{1}{2m-1}abb$.

2°. Sit $2m < 1$. Tunc $S = \frac{1}{1-2m}(yy(a+x) - abb) = \frac{1}{1-2m}(bba^{2m}(a+x)^{1-2m} - abb)$ $= \frac{1}{1-2m}abb\left(\frac{a+x}{a}\right)^{1-2m}$. Quoniam autem nullus eſt limes magnitudinis abſciſſæ

x ſeu

x ſeu $a+x$, nullus etiam limes eſt magnitudinis quantitatis $abb\left(\frac{a+x}{a}\right)^{1-2m}$; & proinde, aucto x, nullus eſt limes magnitudinis ſolidi S.

3°. Sit $2m = 1$. Ex æquatione $S = \frac{1}{1-1}(bba - abb)$ nihil deduci poteſt præ- æqualitatem cylindrorum $ABSD$, $MPSQ$. Æquatio differentialis $\frac{dS}{dx} = \frac{bba}{a+x}$ nos monet, hunc caſum ad logarithmos reduci. (§. 62.)

Scholium. Caſu, quo $2m > 1$, & proinde $S = \frac{1}{2m-1}\left(abb - \frac{bba^{2m}}{(a+x)^{2m-1}}\right)$, ſumta x ex altera parte ordinatæ AB, fit $S = \frac{1}{2m-1}\left(abb - \frac{bba^{2m}}{(a-x)^{2m-1}}\right)$ $= \frac{1}{2m-1}abb\left(1 - \left(\frac{a}{a-x}\right)^{2m-1}\right) = -\frac{1}{2m-1}abb\left(\left(\frac{a}{a-x}\right)^{2m-1} - 1\right)$. Quoniam autem nullus eſt limes parvitatis differentiæ $a-x$, nullus etiam limes eſt magnitudinis quantitatis $\left(\frac{a}{a-x}\right)^{2m-1}$; & proinde nullus eſt limes magnitudinis ſolidi S. Quoniam autem $y = \frac{ba^m}{(a-x)^m}$; facto $x=a$, fieret $y = \frac{ba^m}{a^m o^m} = \frac{b}{o^m}$; unde (Cap. IX.) impoſſibile eſt, ut ſit $x = a$: & in æquatione $S = -\frac{1}{2m-1}abb\left(\left(\frac{a}{a-x}\right)^{2m-1} - 1\right)$ poſito $x = a$, fit $S = -\frac{1}{2m-1}abb\left(\frac{1}{o^{m-1}} - 1\right)$; quo rurſus monemur, impoſſibile eſſe, aſſignare ſolidum, etiam nunc fictum, ſpatio etiamnum ficto abſciſſæ BS reſpondens.

Cum omnia ratiocinia, expreſſioni $S = -\frac{1}{2m-1}abb\left(\frac{1}{o^{2m-1}} - 1\right)$ ad varios infinitorum ordines illuſtrandos ſuperſtructa, contradictoria nitantur ſuppoſitione, poſſe fieri $x = a$; conſequentias inde ductas labi neceſſe eſt.

Idem dicatur de altero caſu, quo $2m < 1$, & proinde $S = \frac{1}{1-2m}abb\left(\left(\frac{b}{y}\right)^{\frac{1-2m}{m}} - 1\right)$; unde ſub contradictoria ſuppoſitione $y = o = b \cdot o$, fit $S = \frac{1}{1-2m}abb\left(\left(\frac{1}{o}\right)^{\frac{1-2m}{m}} - 1\right)$.

§. 113. Determinatio capacitatis ſolidorum rotundorum nonnunquam etiam facilius peragitur modo ſequenti.

Baſi figuræ genitricis in partes quotcunque æquales diviſa, curvæ inſcribantur

bantur & circumſcribantur rectangula, quorum reliqua latera ſint axi revolutionis parallela. Rectangula hæc gyratione ſua circa axem gignent annulos cylindricos, ſolido inſcriptos & circumſcriptos. Solidum rotatione curvæ genitum majus eſt ſumma annulorum cylindricorum rectangulis inſcriptis genitorum; minus vero ſumma annulorum cylindricorum rectangulis circumſcriptis genitorum: ſed idem ſolidum limes eſt tam prioris ſummæ creſcentis, quam poſterioris ſummæ decreſcentis.

Sit *SMAB* ſegmentum curvæ, quod rotatione circa axem *SB* baſi *AB* perpendicularem gignit aliquod ſolidum; & ſegmento huic circumſcribatur rectangulum *SBAD*, quod ſua circa eundem axem rotatione gignet cylindrum ſolido circumſcriptum. Dividatur *AB* in partes quotcunque æquales: ſit RR' una harum partium. Ducantur RM, $R'M'$ ipſi *AB* perpendiculares, quæ curvæ in M, M' occurrant, & ipſi *SD* in P, P'; ducantur Mm, $M'm'$ ipſi *AB* parallelæ, quæ rectis $M'R'$, MR in m & m' reſpective occurrant. Fig. 4.

Annuli cylindrici, gyratione rectangulorum RM', $R'M$, RP' geniti, ſunt inter ſe reſpective uti ipſa rectangula RM', $R'M$, RP'; ſeu uti lineæ $M'R'$, MR, RP. Solidum rotatione curvæ *SMA* genitum dicatur S, & cylindrus rotatione rectanguli *ABSD* genitus dicatur C, ac mutationes ſimultaneæ horum ſolidorum dicantur ΔS, ΔC; erit $\Delta S : \Delta C \begin{matrix} > MR : SB \\ < M'R' : SB \end{matrix}$. Sed prima ratio poteſt fieri $\begin{matrix}\text{minor}\\ \text{major}\end{matrix}$ quacunque ratione propoſita, quæ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ ſit ratione $\begin{matrix} MR : SB \\ M'R' : SB \end{matrix}$. Proinde (§. 1.)

$$\lim.\Delta S : \Delta C = RM : SB.$$

Atqui $$\Delta C : C = (BR+BR')RR' : BA^2; \text{ proinde}$$

$$\lim.\Delta C : C = 2BR.RR' : BA^2; \text{ hinc}$$

$$\lim.\Delta S : C = 2BR.RR'.RM : SB.BA^2$$

$$= 2\pi.BR.RR'.RM : SB.BA^2.\pi$$

Facto igitur $BR = x$, $RR' = \Delta x$, $RM = y$,

eſt $$\lim.\frac{\Delta S}{\Delta x} : 2\pi.xy = C : SB.BA^2.\pi$$

$$= 1 : 1:$$

unde $\frac{dS}{dx} = 2\pi xy$; quæ igitur eſt æquatio differentialis ſolidi, ſpatio *SMAB* circa axem *SB* rotato geniti.

Exemplum 1. Figura *SAB* ſit triangulum rectilineum. Sit $AB = b$: erit $SB = b\,\text{tang.}A$, $y = (b - x)\,\text{tang.}A$, $\frac{dS}{dx} = 2\pi x(b - x)\,\text{tang.}A$, $S = C + \pi(bxx - \frac{2}{3}x^3)\,\text{tang.}A$. Solidum autem evaneſcit, quando $x = 0$; ergo $C = 0$, $S = \pi x\,\text{tang.}A(bx - \frac{2}{3}xx) = SB(bx - \frac{2}{3}xx)\pi$. Sit $x = b$; fit $S = \frac{1}{3}\pi.SB.BA^2$ (uti notum).

Exemplum 2. Figura *SMAB* ſit quadrans circuli, cujus centrum *B*, & radii BS, $BA = r$; erit $y = \sqrt{(rr - xx)}$, $\frac{dS}{dx} = 2\pi x\sqrt{(rr - xx)}$, $S = \pi(C - \frac{2}{3}(rr - xx)^{\frac{3}{2}})$. Sit $x = 0$; fit $S = \pi(C - \frac{2}{3}r^3)$: proinde $C = \frac{2}{3}r^3$, & $S = \pi(\frac{2}{3}r^3 - \frac{2}{3}(rr - xx)^{\frac{3}{2}})$. Sit $x = r$; tum $S = \frac{2}{3}r^3.\pi$ (uti notum).

Exemplum 3. Figura *SMAB* ſit parabola conica, circa axem *SB* rotata. Fit $y = \frac{bb - xx}{p}$, $\frac{dS}{dx} = 2\pi\left(\frac{bbx - x^3}{p}\right)$, $S = 2\pi\left(\frac{(\frac{1}{2}bbxx - \frac{1}{4}x^4)}{p}\right) = \frac{bbxx - \frac{1}{2}x^4}{p}\pi$. Sit $x = b$, erit $S = \pi.\frac{1}{2}\frac{x^4}{p} = \frac{1}{2}SB.AB^2.\pi$.

§. 114. Quoniam paraboloidum & hyperboloidum cubaturæ magni ſunt in determinanda ſolidorum rotundorum capacitate momenti; eaſdem paulo aliter explicare e re eſſe cenſeo.

Fig. 24. Sit $SMM'P'$ dimidium ſegmentum parabolæ, cujus æquatio eſt $y^{m+n} = a^m x^n$. Per verticem *S* ducatur tangens *SA*. Segmento *SMP* & ſpatio exteriori *ASM* inſcribantur v. gr. rectangula $MPP'm$, $MQQ'm'$. Spatio $SP'M'Q'$ circa axem *SP* revoluto, rectangulum $MMP'm$ gignit cylindrum ſegmento paraboloidico inſcriptum; & rectangulum $MQQ'm'$ gignit annulum cylindricum ſolido, quod rotatione ſpatii exterioris $SM'Q'$ gignitur, inſcriptum.

Atqui ratio cylindri $MPP'm$ ad annulum $MQQ'm'$ eadem eſt, quæ ſolidorum $Mm \times MP^2$, $Mm' \times QM(2MP \times Mm')$; & proinde limes rationis cylindri $MPP'm$ & annuli $MQQ'm'$ æqualis eſt limiti rationis ſolidorum $Mm \times MP^2$ & $Mm' \times MQ(2MP + Mm')$.

Sed

$$\begin{array}{llcll}
\text{Sed} & \lim. MP & : & 2MP+Mm' & = 1 : 2 \\
& \lim. Mm & : & Mm' & = PT : MP \quad (\S. 40.) \\
& MP & : & QM & = MP : SP \\
\text{Ergo} & \lim. MP^2 \times Mm & : & Mm'. QM(2MP+Mm') & = PT : 2SP \quad (\S. 14.) \\
& & & & = m+n : 2n \quad (\S. 42.)
\end{array}$$

Proinde $\lim. \text{cyl.} MPP'm : \text{ann. cyl.} MQQ'm' = m+n : 2n.$ Proinde etiam (§. 15.) limes rationis ſummæ omnium cylindrorum paraboloidi inſcriptorum, ad ſummam omnium annulorum cylindricorum ſolido exteriori *MSQ* inſcriptorum, æqualis eſt eidem rationi conſtanti $m+n : 2n$. Unde etiam (§. 4.) ratio limitum harum ſummarum, nempe ratio paraboloidis *SMP* ad ſolidum exterius *SQM*, æqualis eſt eidem rationi datæ. Hinc paraboloides *SMP* eſt ad cylindrum *SPMQ*, uti $m+n$ ad $m+3n$.

2°. Spatium hyperbolicum *ABPM*, cujus æquatio eſt $y^m(a+x)^n = b^m a^n$, circa aſymptotum *SP* revolvatur. Circumſcribantur v. gr. rectangula $MPP'm$, $mQQ'M'$. Cylindrus rectangulo $MPP'm$, & annulus cylindricus rectangulo $mQQ'M'$ geniti, ſunt inter ſe uti ſolida $PP' \times MP^2$, $M'm . Q'M'(2MP-M'm)$. Proinde limes rationis cylindri hujus & annuli cylindrici æqualis eſt limiti rationis horum ſolidorum. Fig. 25.

$$\begin{array}{llcll}
\text{Atqui} & \lim. MP & : & 2MP-M'm & = 1 : 2 \\
& \lim. PP' & : & M'm & = PT : MP \quad (\S. 40.) \\
& \lim. MP & : & M'Q' & = MP : MQ \ (=SP) \\
\text{Ergo} & \lim. MP^2 . PP' & : & M'm . M'Q'(2MP-Mm) & = PT : 2SP \quad (\S. 14.) \\
& & & & = m : 2n \quad (\S. 42.) \\
\text{Hinc} & \lim.\text{cyl.} MPP'm & : & \text{ann. cyl.} mQQ'M' & = m : 2n.
\end{array}$$

Proinde etiam ratio ſummæ omnium cylindrorum, ſolido rotatione ſpatii *AMPB* genito circumſcriptorum, ad ſummam omnium annulorum cylindricorum, ſolido rotatione ſpatii *AMQD* genito circumſcriptorum, æqualis eſt eidem rationi conſtanti $m : 2n$: & proinde ſolida, quæ limites ſunt harum ſummarum, & quæ gignuntur rotatione ſpatiorum *AMPB*, *AMQD*, ſunt in eadem ratione data $m : 2n$ (§. 4.)

1°. Sit

1°. Sit $m > 2n$. Erit etiam ſol. $AMPB >$ ſol. $AMQD$, & ſol. $AMPB$–ſol. $AMQD$ $=$ cyl. $MPSQ$ — cyl. $ABSD$; proinde ſol. $AMBP$: cyl. $MPSQ$ — cyl. $ABSD$ $= m : m - 2n$.

2°. Sit $m < 2n$: erit ſol. $AMQD >$ ſol. $AMPB$, & ſol. $AMQD$—ſol. $AMPB$ $=$ cyl. $ABSD$—cyl. $MPSQ$; proinde ſol. $AMPB$: cyl. $ABSD$—cyl. $MPSQ$ $= m : 2n - m$.

3°. Sit $m = 2n$. Nihil ulterius inde concludi poteſt, niſi quod ſolida $AMPB$, $AMQD$ ſint inter ſe æqualia; & proinde etiam cylindri $MPSQ$, $ABSD$ ſint inter ſe æquales.

§. 115. Curva generatrix ſolidi referatur ad focum aliquem per radios vectores ad hunc focum ductos, & per angulos, quos radii vectores cum recta poſitione data comprehendunt. Capacitas ſolidi determinabitur modo ſequenti.

Fig. 26. Sit $SM'M$ curva ad focum F relata per radios vectores FM, FM', & angulos SFM, SFM', quos radii vectores cum recta FS poſitione data comprehendunt. Centro F radio FS (pro unitate ſumto) deſcribatur circulus, qui radiis vectoribus FM, FM' in punctis X, X' occurrat. Tum centro F radiis FM, FM' deſcriptis arcubus Mm, $M'm'$, inſcribantur & circumſcribantur curvæ ſectores circulares MFm, $M'Fm'$. Solida ſectoribus his circa axem SF revolutis genita ſunt inter ſe in ratione triplicata radiorum FM, FM'. Sed ratio æqualitatis limes eſt rationis horum radiorum: proinde & ratio æqualitatis limes eſt rationis ſolidorum ab his ſectoribus genitorum (§. 4.); tantoque magis ratio æqualitatis limes eſt rationis alterutrius horum ſolidorum ad ſolidum rotatione ſectoris curvæ MFM' genitum.

Sit $Fm = y$, $FM' = y'$, $FS = r$, angulus $SFX = x$, $XFX' = \Delta x$. Notum eſt, ſolidi rotatione ſectoris circularis XFX' geniti expreſſionem eſſe $\frac{2}{3}\pi r^3(\text{coſ.}x' - \text{coſ.}x) = \frac{2}{3}\pi r^3 \Delta \text{coſ.}x$. Sit S ſolidum ſectore curvæ SFM genitum; eſt

eſt	lim. ſol. MFM'	: ſol. MFm	=	I	:	I	
	ſol. MFm	: ſol. XFX'	=	y^3	:	r^3	
hinc	lim. ſol. MFM'	: ſol. XFX'	=	y^3	:	r^3	(§. 14.)
ſeu	lim. ΔS	: $\frac{2}{3}\pi r^3 \Delta \text{coſ.}x$	=	y^3	:	r^3	

Atqui

Atqui $\lim.\Delta\,\text{coſ.}x : \Delta x = \text{ſin.}x : 1$ (§. 76.)

Ergo $\lim.\Delta S : \frac{2}{3}\pi r^3 \Delta x = y^3\,\text{ſin.}x : r^3$

et $\lim.\frac{\Delta S}{\Delta x} : \frac{2}{3}\pi r^3 = y^3\,\text{ſin.}x : r^3$

$= \frac{2}{3}\pi y^3\,\text{ſin.}x : \frac{2}{3}\pi r^3$;

proinde $\lim.\frac{\Delta S}{\Delta x} = \frac{2}{3}\pi y^3\,\text{ſin.}x$

ſeu $\frac{dS}{dx} = \frac{2}{3}\pi y^3\,\text{ſin.}\,x$, quæ eſt æquatio differentialis ſolidi propoſiti.

Exempla. 1°. Sit y quantitas data $= r$: $\frac{dS}{dx} = \frac{2}{3}\pi r^3\,\text{ſin.}x$, $S = C - \frac{2}{3}\pi r^3\,\text{coſ.}x$, Sit $S = 0$, quando $x = 0$: $C = \frac{2}{3}\pi r^3$, $S = \frac{2}{3}\pi r^3(1 - \text{coſ.}x) = \frac{4}{3}\pi r^3\,\text{ſin.}^2\tfrac{1}{2}x$. Sit $x = 180°$: $S = \frac{4}{3}\pi r^3$.

2°. Sit $y = a\,\text{ſec.}\,x$ (quæ eſt æquatio ad lineam rectam): $\frac{dS}{dx} = \frac{2}{3}\pi a^3(\text{ſec.}^3 x\,\text{ſin.}\,x = \frac{2}{3}\pi a^3(\text{ſec.}^2 x\,\text{tang.}\,x)$, $S = \frac{2}{3}\pi a^3(C + \frac{1}{2}\text{ſec.}^2 x)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{2}$, $S = \frac{1}{3}\pi a^3\,\text{tang.}^2 x$.

3°. Sit $y = a\,\text{ſec.}^2\tfrac{1}{2}x$ (quæ eſt æquatio parabolæ conicæ): $\frac{dS}{dx} = \frac{2}{3}\pi a^3\,\text{ſec.}^6\tfrac{1}{2}x\,\text{ſin.}x = \frac{4}{3}\pi a^3\,\text{ſec.}^4\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x$, $S = \frac{4}{3}\pi a^3(C + \frac{1}{2}\text{ſec.}^4\tfrac{1}{2}x)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{2}$, $S = \frac{2}{3}\pi a^3(\text{ſec.}^4\tfrac{1}{2}x - 1) = \frac{2}{3}\pi a^3(\text{ſec.}^2\tfrac{1}{2}x + 1)(\text{ſec.}^2\tfrac{1}{2}x - 1)$ $= \frac{2}{3}\pi a^3\left(\left(\frac{y}{a} + 1\right)\left(\frac{y}{a} - 1\right)\right) = \frac{2}{3}\pi a(y + a)(y - a) = \frac{2}{3}\pi a(yy - aa)$.

4°. Sit $y = \frac{bb}{a + e\,\text{coſ.}\,x}$ (quæ eſt æquatio ellipſeos conicæ): $\frac{dS}{dx} = \frac{2}{3}\pi\left(\frac{b^6}{(a + e\,\text{coſ.}x)^3}\,\text{ſin.}x\right)$, $S = \frac{1}{3}\pi\left(C + \frac{b^6}{e(a + e\,\text{coſ.}x)^2}\right)$. Sit $S = 0$, quando $x = 0$: $C = -\frac{b^6}{e(a+e)^2}$; $S = \frac{1}{3}\pi\left(\frac{b^6}{e(a + e\,\text{coſ.}x)^2} - \frac{b^6}{e(a+e)^2}\right) = \frac{1}{3}\pi\left(\frac{bbyy}{e} - \frac{bb(a-e)^2}{e}\right)$ $= \frac{1}{3}\pi\frac{bb(yy - (a-e)^2)}{e}$.

§. 116. Series Bernoulliana applicari etiam poteſt determinationi capacitatis ſolidorum rotundorum.

1°. Quoniam eſt (§. 112.) $\frac{dS}{dx} = \pi . yy$; fit per §. 36.

 $S = \pi$

$$S = \pi \Big(xyy$$
$$- 2 . \frac{x^2}{1.2} . y \frac{dy}{dx}$$
$$+ 2 . \frac{x^3}{1.2.3} \Big(y \frac{ddy}{dx^2} + \Big(\frac{dy}{dx}\Big)^2 \Big)$$
$$- 2 . \frac{x^4}{1.2...4} \Big(y \frac{d^3y}{dx^3} + 3 \frac{dy}{dx} . \frac{ddy}{dx^2} \Big)$$
$$+ 2 . \frac{x^5}{1.2...5} \Big(y \frac{d^4y}{dx^4} + 4 \frac{dy}{dx} . \frac{d^3y}{dx^3} + 3 \Big(\frac{ddy}{dx^2}\Big)^2 \Big)$$
$$- 2 . \frac{x^6}{1.2...6} \Big(y \frac{d^5y}{dx^5} + 5 \frac{dy}{dx} . \frac{d^4y}{dx^4} + 10 \frac{ddy}{dx^2} . \frac{d^3y}{dx^3} \Big)$$

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Quæ ſeries, utut regularis, non ſemper eſt calculo omnium promtiſſimo accommodata; imo & non abrumpitur quibusdam etiam caſibus, quibus alia via capacitas ſolidi terminis numero finitis exprimi poteſt.

2°. Quoniam eſt (§. 113.) $\frac{dS}{dx} = 2\pi xy$; erit (§. 36.)

$$S = 2\pi \Big(\frac{1}{1.2} xxy - \frac{x^3}{1.2.3} . \frac{dy}{dx} + \frac{x^4}{1...4} \frac{ddy}{dx^2} - \frac{x^5}{1...5} \frac{d^3y}{dx^3} + \frac{x^6}{1...6} \frac{d^4y}{dx^4} - \ldots \Big)$$

Cylindrorum ſolido rotundo tam inſcriptorum quam circumſcriptorum ſummæ eodem modo determinari poſſunt, quo rectangulorum ſpatio curvilineo inſcriptorum & circumſcriptorum ſummæ §. 103. determinatæ fuerunt.

§. 117. Solidorum rotundorum cubatura reduci poteſt ad rectificationem circuli & ad quadraturam alterius curvæ.

Etenim quoniam eſt (§. 112.) $\frac{dS}{dx} = \pi yy$; ſi deſcribatur ſuper eodem axe, & iisdem axis abſciſſis, altera curva, cujus ordinatæ y' ſint quadratis ordinatarum curvæ genitricis proportionales; ita ut $ay' = yy$: erit $\frac{dS}{dx} = \pi a y'$, & $\frac{dS'}{dx} = y'$; ideoque capacitas prioris ſolidi erit areæ poſterioris curvæ proportionalis.

Porro quoniam eſt (§. 113.) $\frac{dS}{dx} = 2\pi xy$: ſi fiat ſemper $ay' = 2xy$; capaci

ta

tas ſolidi erit areæ curvæ, cujus æquatio differentialis eſt $\frac{dS'}{dx} = y'$ proportionalis.

§. 118. Data expreſſione capacitatis alicujus ſolidi rotundi per rectas axi ordinatim applicatas, determinari poteſt natura curvæ genitricis.

Exempla. 1°. Sit $S = \pi aax$; ideoque $\frac{dS}{dx} = \pi aa$: hinc $yy = aa$, $y = a$; proinde baſis eſt conſtans, & ſolidum propoſitum eſt cylindrus rectus.

2°. Sit $S = \pi \frac{x^m}{a^{m-3}}$: hinc $\frac{dS}{dx} (= \pi yy) = m\pi \frac{x^{m-1}}{a^{m-3}}$; $yy = m\frac{x^{m-1}}{a^{m-3}}$.

3°. Sit $S = \pi . \frac{m}{n} xyy$: $\frac{dS}{dx} (= \pi yy) = \pi . \frac{m}{n} yy + 2\pi xy \frac{dy}{dx}$; $yy = \frac{m}{n} yy + 2xy \frac{dx}{dy}$;
$y = \frac{m}{n} y + 2x \frac{dy}{dx}$; $2x \frac{dy}{dx} = \frac{n-m}{n} . y$; $y^{2n} = C^{n+m} x^{n-m}$.

Caput decimum tertium.

De ſuperficiebus ſolidorum rotundorum.

§. 119.

Linea curva quæcunque circa axem aliquem rotata generet ſuperficiem curvam. Dico: rationem æqualitatis limitem eſſe ſuperficierum a chorda & ab arcu curvæ ſimul genitarum.

Sit SMM' curva, quæ circa axem SP rotata ſuperficiem curvam generat. Fig. 31. Dico: rationem æqualitatis limitem eſſe rationis ſuperficierum a chorda MM & ab arcu MZM ſimul genitarum.

Per M & M' actæ concipiantur rectæ Mm & $M'm'$ axi SP parallelæ & arcui MM' æquales, quæ ſimul cum curva rotatæ ſuperficies curvas cylindricas gignent. Quoniam ſingulæ partes rectæ Mm axi SP parallelæ minus ab hoc axe diſtant, quam pars quælibet arcus MZM'; ſuperficies a recta Mm genita minor eſt ſuperficie ab arcu MZM' ſimul genita: contra quoniam ſingulæ partes rectæ $M'm'$ magis diſtant ab axe, quam pars quælibet arcus MZM'; ſuperficies a recta $M'm'$ genita major eſt ſuperficie ab MZM' ſimul genita. Atqui ſuperficies cylindricæ rotatione rectarum Mm, $M'm'$ genitæ ſunt inter ſe uti or-

dinatæ MP, $M'P'$; & proinde ratio æqualitatis limes eſt rationis harum ſuperficierum; & a fortiori ratio æqualitatis limes eſt rationis alterutrius harum ſuperficierum ad ſuperficiem rotatione arcus genitam.

Scilicet arcu MZM' poſito $= \Delta z$, & ſuperficie ab eo genita $= \Delta S$; eſt $\lim. \Delta S : 2\pi . y\Delta z = 1 : 1$.

Superficies rotatione chordæ & rectæ Mm genitæ ſunt inter ſe uti rectangula $\sqrt{(\Delta x^2 + \Delta y^2)}\frac{2y+\Delta y}{2}$ & $y\Delta z$.

$$\text{Atqui} \quad \lim.\sqrt{(\Delta x^2+\Delta y^2)} : \Delta z = 1 : 1 \quad (\S.\ 47.)$$

$$\lim.\frac{2y+\Delta y}{2} : y = 1 : 1$$

$$\text{ergo} \quad \lim.\sqrt{(\Delta x^2+\Delta y^2)}.\frac{2y+\Delta y}{2} : y\Delta z = 1 : 1 \quad (\S.\ 14.)$$

$$\text{Sed} \quad \lim. y\Delta z : \Delta S = 1 : 2\pi$$

$$\text{ergo} \quad \lim.\sqrt{(\Delta x^2+\Delta y^2)}.\frac{2y+\Delta y}{2} : \Delta S = 1 : 2\pi \quad (\S.\ 14.)$$

$$\text{ſeu} \quad \lim.\sqrt{\left(1+\frac{\Delta y^2}{\Delta x^2}\right)}.\frac{2y+\Delta y}{2} : \frac{\Delta S}{\Delta x} = 1 : 2\pi$$

$$\text{ſeu} \quad \lim.\frac{\Delta S}{dx} = 2\pi \lim.\sqrt{\left(1+\frac{\Delta^2}{\Delta x^2}\right)}.\frac{2y+\Delta y}{2};$$

et proinde $\frac{dS}{dx} = 2\pi y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$; quæ eſt æquatio differentialis ſuperficiei rotatione curvæ genitæ.

1°. Arcus MZM' totus ſit concavus verſus axem rotationis SP. Quoniam arcus major eſt chorda; & partes arcus inter rectas quaslibet axi ordinatim applicatas ab axe rotationis magis diſtant, quam ab eodem axe diſtant partes chordæ ordinatis iisdem interjacentes: duplici hoc reſpectu ſuperficies rotatione arcus genita major eſt ſuperficie, quæ rotatione chordæ gignitur.

2°. Quodſi autem arcus eſt verſus axem rotationis convexus; generatim de æqualitate aut inæqualitate ſuperficierum ab arcu & chorda ſimul genitarum ſtatui nihil poteſt. Nempe quatenus partes arcus duas inter quaslibet axi ordinatim applicatas contentæ huic viciniores ſunt, quam partes chordæ ordinatis iisdem interjacentes; ſuperficies arcu genita minor fit ſuperficie a chorda genita: dum ex altera parte ob arcum chorda majorem prior ſuperficies major

eſt

eſt poſteriore. Quare hoc caſu æquatio differentialis $\frac{dS}{dx} = 2\pi y \sqrt{}(1 + (\frac{dy}{dx})^2)$, limitum notione, prouti §. 13. fuit extenſa, nititur.

§. 120. *Exemplum* 1. Sit $yy = rr - xx$: $\frac{dy}{dx} = -\frac{x}{y}$, $\sqrt{}(1 + (\frac{dy}{dx})^2) = \frac{r}{y}$, $\frac{dS}{dx} = 2\pi r$, $S = 2\pi r x$. Nempe ſuperficies ſphæricæ inter duo plana ſibi invicem parallela comprehenſæ ſunt inter ſe uti diſtantiæ horum planorum.

Exemplum 2. Sit $y = x \,\text{tang.}\varphi$ (quæ eſt æquatio lineæ rectæ): $\frac{dS}{dx} = 2\pi x \,\text{tang.}\varphi \,\text{ſec.}\varphi$, $S = \pi x x \,\text{tang.}\varphi \,\text{ſec.}\varphi = \pi x y \,\text{ſec.}\varphi$.

Exemplum 3. Sit $yy = 2px$ (quæ eſt æquatio parabolæ): $\frac{dS}{dx} = 2\pi \sqrt{}(pp + yy)$ $= 2\pi \sqrt{}(pp + 2px)$, $S = 2\pi (C + \frac{1}{3p}(pp + 2px)^{\frac{3}{2}})$. Sit $S = 0$, quando $x = 0$: $C = -\frac{1}{3}pp$, $C = 2\pi(\frac{1}{3p}(pp + 2px)^{\frac{3}{2}} - \frac{1}{3}pp) = \frac{2}{3}\pi p((p + 2x)\sqrt{\frac{p+2x}{p}} - p)$.

Exemplum 4. Sit $yy = \frac{bb}{aa}(aa - xx)$.

1°. Sit $b = a$: $\frac{dS}{dx} = 2\pi a$ (quæ eſt æquatio differentialis ſuperficiei ſphæricæ).

2°. Sit $b < a$, ſeu ellipſis gyretur circa axem transverſum; ſit $aa - bb = ee$: $\frac{dS}{dx} = 2\pi \frac{be}{aa} \sqrt{}(\frac{a^4}{ee} - xx)$, $S = \pi(C + \frac{aab}{e}(\text{arc.ſin.}\frac{xe}{aa} + \frac{ex}{aa}\sqrt{}(1 - \frac{ee}{aa} \cdot \frac{xx}{aa})))$. Sit $S = 0$, quando $x = 0$: $C = 0$. Sit $x = a$: $S = \pi \frac{aab}{e}(\text{arc.ſin.}\frac{e}{a} + \frac{eb}{aa})$ $= \pi(bb + \frac{aab}{e}\text{arc.ſin.}\frac{e}{a})$.

Obſervatio. $\text{Arc.ſin.}\frac{e}{a} = \text{arc.tang.}\frac{e}{b} = \frac{e}{b} - \frac{1}{3} \cdot \frac{e^3}{b^3} + \frac{1}{5} \cdot \frac{e^5}{b^5} - \frac{1}{7} \cdot \frac{e^7}{b^7} + \ldots$ (§. 79.)

Hinc $\frac{aab}{e}\text{arc.ſin.}\frac{e}{a} = aa(1 - \frac{1}{3}\frac{ee}{bb} + \frac{1}{5}\frac{e^4}{b^4} - \frac{1}{6}\frac{e^6}{b^6} + \ldots)$

Proinde facto $e = 0$, $S = \pi(bb + aa) = 2bb\pi$, juxta Ex. 1.

3°. Ellipſis gyretur circa axem ſecundum; unde $yy = \frac{aa}{bb}(bb - xx)$: $\frac{dS}{dx} = 2\pi \cdot \frac{ae}{bb}\sqrt{}(\frac{b^4}{ee} + xx)$, cujus integratio reducitur ad aream hyperbolæ conicæ ſeu ad logarithmos. Scilicet

$S = \pi$

$S = \pi \frac{ae}{bb}(C + x\sqrt{(\frac{b^4}{ee} + xx)} + \frac{b^4}{ee}\log.(\sqrt{\frac{b^4}{ee} + xx}) + x)$. Sit $S = 0$, quando $x = 0$:

$C = -\log.\frac{bb}{e}$, $S = \pi.\frac{ae}{bb}\left(x\sqrt{(\frac{b^4}{ee} + xx)} + \frac{b^4}{ee}\log.\frac{\sqrt{(\frac{b^4}{ee} + xx)} + x}{\frac{bb}{e}}\right]$. Sit $x = b$:

$$S = \pi.\frac{ae}{bb}\left(b \times \frac{ab}{e} + \frac{b^4}{ee}\log.\frac{\frac{ab}{e} + b}{\frac{bb}{e}}\right]$$

$$= \pi\left(aa + \frac{abb}{e}\log.\frac{a+e}{b}\right) = \pi\left(aa + \frac{abb}{e}\log.\frac{b}{a-e}\right) = \pi\left(aa + \frac{abb}{e}\log.\sqrt{\frac{a+e}{a-e}}\right).$$

Obſervatio. $\log.\sqrt{\frac{a+e}{a-e}} = \frac{e}{a} + \frac{1}{3}.\frac{e^3}{a^3} + \frac{1}{5}.\frac{e^5}{a^5} + \ldots$ (§. 61.)

hinc $\frac{abb}{e}\log.\sqrt{\frac{a+e}{a-e}} = ab(1 + \frac{1}{3}.\frac{ee}{aa} + \frac{1}{5}.\frac{e^4}{a^4} + \ldots)$

proinde facto $e = 0$, $S = \pi(aa + ab) = 2\pi aa$, juxta Ex. 1.
Superficies hyperboloidum, rotatione hyperbolæ circa alterutrum axem genitæ, eodem modo determinantur.

Exemplum 5. Sit $y(a+x) = ab$, ſeu hyperbola conica gyretur circa aſymptotum. $\frac{dS}{dx} = 2\pi\frac{ab}{a+x}\sqrt{(1 + \frac{aabb}{(a+x)^4})}$: unde facto $S = 0$, quando $x = 0$;

$$S = \pi.ab\left(\sqrt{(1 + \frac{bb}{aa})} - \sqrt{(1 + \frac{aabb}{(a+x)^4})} + \log.\frac{\sqrt{((aa+bb)} - a}{\sqrt{(\frac{(a+x)^4}{aa} + bb)} - \frac{(a+x)^2}{a}}\right].$$

Exemplum 6. Sit $y = \sqrt{(2rx - xx)} + \text{arc.ſin.v.}\frac{x}{r}$, quæ eſt æquatio cycloidis vulgaris. Fit $\frac{dS}{dx} = 2\pi(\sqrt{(2r(2r-x)} + r\sqrt{\frac{2r}{x}}\,\text{arc.ſin.v.}\frac{x}{r})$:

unde $S = 2\pi(C - \frac{2}{3}(2r-x)\sqrt{2r(2r-x)} + 4r\sqrt{2r(2r-x)} + 2r\sqrt{2rx}\,\text{arc.ſin.v.}\frac{x}{r})$.
Sit $S = 0$, quando $x = 0$: $C = -\frac{16}{3}rr$. Sit $x = 2r$: $S = 8\pi rr(\pi - \frac{4}{3})$.

§. 121. Quoniam $\frac{dS}{dx} = 2\pi y\sqrt{(1 + (\frac{dy}{dx})^2)}$; & normalis expreſſio eſt $y\sqrt{(1 + (\frac{dy}{dx})^2}$ (§. 110.): quadratura ſuperficiei ſolidorum rotundorum pendet ab

ab normali curvæ genitricis. Scilicet ſuper axe curvæ genitricis deſcribatur curva talis, ut rectæ axi ordinatim applicatæ æquales ſint (aut proportionales) normalibus curvæ genitricis iisdem abſciſſis reſpondentibus. Superficies ſolidi rotatione prioris curvæ geniti proportionalis erit areæ curvæ poſterioris. Ratio autem harum ſuperficierum ea eſt, quæ circumferentiæ circuli & radii.

Scholium. Seriei Bernoullianæ ad determinandas ſolidorum rotundorum ſuperficies applicandæ non immoror; quoniam exponentes differentiales ordinum ſucceſſivorum quantitatis $y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}$ adeo fiunt compoſiti, ut nulla inde utilitas duci poſſit.

§. 122. Ratio differentialis ſuperficierum, a curvis ad punctum datum relatis genitarum, eodem modo poteſt determinari. Sit nempe angulus $SFM=z$, & radius vector $FM=r$; eſt $\frac{dS}{dx}=2\pi r\,\text{ſin.}z\sqrt{\left(rr+\left(\frac{dy}{dx}\right)^2\right)}$. Exercitii cauſa oſtendere ſufficiat, quomodo ratio hæc differentialis ex priore deducatur. Eſt ideo Fig. 26.

$$y=r\,\text{ſin.}z;\quad \text{igitur}\ \frac{dy}{dz}=\frac{dr}{dz}\text{ſin.}z+r\,\text{coſ.}z$$

$$x=r\,\text{coſ.}z\qquad \frac{dx}{dz}=\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z$$

$$\frac{dy}{dx}=\frac{\frac{dr}{dz}\text{ſin.}z+r\,\text{coſ.}z}{\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z}$$

$$\text{hinc}\ \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}=\frac{\sqrt{\left(rr+\left(\frac{dr}{dz}\right)^2\right)}}{\frac{dr}{dz}\text{coſ.}z-r\,\text{ſin.}z}=\frac{\sqrt{\left(rr+\left(\frac{dr}{dz}\right)^2\right)}}{\frac{dx}{dz}}=\frac{dr}{dz}\sqrt{\left(rr+\left(\frac{dr}{dz}\right)^2\right)}.$$

$$\text{Atqui}\ \frac{dS}{dx}=2\pi y\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)};$$

$$\text{ergo}\ \frac{dS}{dx}=2\pi r\,\text{ſin.}z\frac{dz}{dx}\sqrt{\left(rr+\left(\frac{dr}{dz}\right)^2\right)}$$

$$\text{et}\ \frac{dS}{dz}=2\pi r\,\text{ſin.}z\sqrt{\left(rr+\left(\frac{dr}{dz}\right)^2\right)}.$$

Exem-

Exemplum 1. Sit r datæ magnitudinis: $\frac{dS}{dz} = 2\pi rr\,\text{ſin.}\,z$, $S = 2\pi rr(C - \text{coſ.}\,z)$. Sit $S = 0$, quando $z = 0$: $C = 1$, $S = 2\pi rr(1 - \text{coſ.}\,z) = 4\pi rr\,\text{ſin.}^2\,\frac{1}{2}z$.

Exemplum 2. Sit $r = a\,\text{ſec.}\,z$ (quæ eſt æquatio lineæ rectæ): $\frac{dr}{dz} = a\,\text{ſec.}\,z\,\text{tang.}\,z$, $\frac{dS}{dz} = 2\pi a\,\text{ſec.}\,z\,\text{ſin.}\,z\,\sqrt{(aa\,\text{ſec.}^2 z + aa\,\text{ſec.}^2 z\,\text{tang.}^2 z)} = 2\pi aa\,\text{tang.}\,z\,\text{ſec.}^2 z$, $S = \pi aa(C + \text{ſec.}^2 z)$. Sit $S = 0$, quando $z = 0$: $C = -1$; $S = \pi aa\,\text{tang.}^2 z$.

Exemplum 3. Sit $r = p\,\text{ſec.}^2\,\frac{1}{2}z$ (quæ eſt æquatio focalis parabolæ): $\frac{dr}{dz} = p\,\text{ſec.}^2\,\frac{1}{2}z\,\text{tang.}\,\frac{1}{2}z$, $\frac{dS}{dz} = 4\pi \cdot pp\,\text{ſec.}^3\,\frac{1}{2}z\,\text{tang.}\,\frac{1}{2}z$, $S = C + \frac{8}{3}\pi pp \cdot \text{ſec.}^3\,\frac{1}{2}z = \frac{8}{3}\pi \cdot pp(C + \text{ſec.}^3\,\frac{1}{2}z)$. Sit $S = 0$, quando $z = 0$: $C = -1$, $S = \frac{8}{3}\pi \cdot pp(\text{ſec.}^3\,\frac{1}{2}z - 1) = \frac{8}{3}\pi \cdot pp\left(\left(\frac{r}{p}\right)^{\frac{3}{2}} - \right) = \frac{8}{3}\pi \cdot (r\sqrt{pr} - pp)$.

§. 123. Quemadmodum ſuperficies ſolidorum rotundorum æquatione curvæ genitricis determinatur: ita viciſſim, expreſſione ſuperficiei ſolidorum rotundorum data, erui poteſt æquatio curvæ genitricis; uti paucis exemplis oſtendam.

Exemplum 1. Superficies ſolidi creſcat uti abſciſſa axis. Fit ideo $S = 2\pi rx$, & $\frac{dS}{dx} = 2\pi r = 2\pi y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$; hinc $1 + \left(\frac{dy}{dx}\right)^2 = \frac{rr}{yy}$, $\left(\frac{dy}{dx}\right)^2 = \frac{rr - yy}{yy}$; $\frac{dx}{dy} = \frac{y}{\sqrt{(rr - yy)}}$; $x = C - \sqrt{(rr - yy)}$, quæ eſt æquatio circuli.

Exemplum 2. Superficies creſcat in ratione duplicata abſciſſæ axis. Eſt ideo $S = \frac{2m}{n}\pi \cdot xx$, $\frac{dS}{dx} = \frac{4m}{n}\pi x = 2\pi y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$; $\frac{2m}{n}x = y\sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$. Huic æquationi ſatisfacit (in caſu particulari) linea recta, quando nempe $\frac{dy}{dx}$ eſt ratio conſtans, unde & ratio $x : y$ eſt conſtans.

Exemplum 3. Curva referatur ad aliquem focum, & ſuperficies creſcat in ratione radii vectoris. Eſt ideo $S = 2\pi ar$; $\frac{dS}{dz} = 2\pi a\frac{dr}{dz} = 2\pi r\,\text{ſin.}\,z\sqrt{\left(rr + \left(\frac{dr}{dz}\right)^2\right)}$; unde $a\frac{dr}{dz} = r\,\text{ſin.}\,z\sqrt{\left(rr + \left(\frac{dr}{dz}\right)^2\right)}$, $\left(\frac{dr}{dz}\right)^2(aa - rr\,\text{ſin.}^2 z) = r^4\,\text{ſin.}^2 z$, $\frac{dr}{dz} = \frac{rr\,\text{ſin.}\,z}{\sqrt{(aa - rr\,\text{ſin.}^2 z)}}$, cujus æquationis integrationem me exhibere non poſſe fateor.

Caput decimum quartum.

De regula Guldini *dicta.*

§. 124.

Recta magnitudine data circa axem quemcunque in eodem plano ſitum revolvatur. Superficies cylindrica vel conica, rotatione rectæ hujus genita, æqualis eſt rectangulo ſub hac linea & ſub circumferentia a puncto lineæ hujus medio deſcripta. Proinde ſuperficies, a recta magnitudine data circa varios axes revoluta genitæ, proportionales ſunt diſtantiis puncti ejus medii ab axe revolutionis.

Pariter cylindrus vel annulus cylindricus, a rectangulo circa unum ipſius latus, vel circa rectam uni ex lateribus ejus parallelam revoluto genitus, æqualis eſt prismati, cujus baſis eſt ipſum rectangulum, & cujus altitudo æqualis eſt circumferentiæ, quam centrum figuræ rectanguli hac revolutione percurrit ſeu generat. Solida igitur ab eodem rectangulo, circa varios axes uni laterum ejus parallelos revoluto, genita proportionalia ſunt diſtantiis centri hujus rectanguli ab axe revolutionis.

His exemplis (omnium ſimpliciſſimis) præmiſſis, facile intelligitur, quid ſibi velit regula capite hoc explicanda. Scilicet lineæ vel ſuperficies quæcunque tam ſpecie quam magnitudine datæ circa axem aliquem revolvantur: ſuperficies vel ſolida, rotatione hac genita, ſunt inter ſe in ratione compoſita ex rationibus magnitudinum atque circumferentiarum a puncto quodam, cujus ſitum magnitudo genitrix determinat, deſcriptarum.

Cum regula hæc determinationem capacitatis & ſuperficiei ſolidorum rotundorum (de qua duobus poſtremis capitibus actum fuit) admodum juvet; e re eſſe cenſeo, fundamenta ejus hoc loco exponere.

§. 125. *Lemma.* Sint quotlibet puncta in eodem plano poſitione data, & ſint totidem rectæ magnitudine datæ. Ex omnibus punctis datis demittantur in rectam quamlibet in eodem plano ſitam rectæ perpendiculares. Sumatur ſumma rectangulorum factorum ex his perpendiculis & ex lineis magnitudine datis reſpective. Dico: dari in eodem plano punctum, ex quo ſi in eandem

 rectam

rectam demittatur perpendiculum, rectangulum, ſub hoc perpendiculo & ſub ſumma rectarum magnitudine datarum contentum, æquale ſit ſummæ priorum rectangulorum.

Fig. 32. *Exemplum primum.* Sint duo puncta A, B, a quibus in rectam quamcunque $A'B'$ demittantur perpendicula AA', BB'. Sint etiam duæ rectæ a, b magnitudine datæ. Recta AB in puncto Z ita ſecetur, ut ſit $AZ \times a = BZ \times b$; & a puncto Z in rectam $A'B'$ demittatur perpendiculum ZY. Dico: fore $AA' \times a + BB' \times b = ZY(a+b)$.

Conſtr. Per Z agatur recta ab ipſi $A'B'$ parallela, quæ rectis AA', BB' in a & b occurrat.

Dem. Propter ſimilia triangula AZa, BZb eſt $Aa : Bb = AZ : BZ = b : a$; hinc $a(ZY - AA') = b(BB' - ZY)$; proinde $ZY(a+b) = a \times AA' + b \times BB'$.

Fig. 33. *Exemplum ſecundum.* Sint tria puncta A, B, C poſitione data, & tres rectæ a, b, c magnitudine datæ; & ſint AA', BB', CC' perpendicula in rectam quamcunque demiſſa.

Puncto Z reſpectu punctorum A, B ita determinato, ut ſit $ZY(a+b) = AA' \times a + BB' \times b$; determinetur punctum Z' reſpectu punctorum Z, C, ita ut $Z'Y'(a+b+c) = ZY(a+b) + CC' \times c$: erit $Z'Y'(a+b+c) = AA' \times a + BB' \times b + CC' \times c$.

Exemplum tertium. Sint quatuor puncta A, B, C, D poſitione data, & quatuor lineæ a, b, c, d magnitudine datæ.

Determinetur punctum Z' reſpectu punctorum A, B, C (juxta exemplum ſecundum); tum determinetur punctum Z'' reſpectu punctorum Z', D, ſic, ut $Z''Y''(a+b+c+d) = Z'Y'(a+b+c) + DD' \times d$: erit $Z''Y''(a+b+c+d) = AA' \times a + BB' \times b + CC' \times c + DD' \times d$.

Ex his exemplis patet methodus generalis procedendi ac demonſtrandi. Propoſitione nimirum evicta pro punctis quotlibet n poſitione datis, & pro totidem rectis magnitudine datis; eadem etiam vera eſſe demonſtratur, ſi tam punctorum quam rectarum numerus unitate augeatur. Cum igitur vera ſit propoſitio de paucis punctis & rectis ,2, 3, 4 . . . .; ea etiam vera eſt pro quinque punctis: inde pro ſex, & ſic deinceps. Ideo ſemper determinari poteſt punctum Z tale, ut ſit $a \times AA' + b \times BB' + c \times CC' + \ldots, l \times LL' = ZZ'(a+b++c+\ldots+l)$.

Scholium.

Scholium. Quoniam puncti Z in plano poſitione dato ſitus determinatur perpendiculis, quæ in duas rectas poſitione datas, ſe mutuo ſecantes, ab eo ducuntur; demonſtrandum eſt: quod, ſi præcedens propoſitio locum habeat pro ejusmodi duabus rectis, eadem quoque valeat pro alia quacunque recta.

Et primo quidem prædicta æquatione locum habente pro recta qualibet poſitione data, eadem valet pro quavis recta priori parallela; cum utrumque æquationis prioris membrum ita augeatur vel minuatur rectangulo ſub diſtantia duarum parallelarum & ſub ſumma $a+b+c+d+\ldots+l$.

2°. Sint duæ rectæ SN, SN' ſibi invicem perpendiculares; & demonſtratum fuerit duas æquationes ſequentes locum habere: Fig. 34.

$$a\times AA'+b\times BB'+c\times CC'+\ldots l\times LL' = ZZ'(a+b+c+d+\ldots l)$$
$$a\times SA'+b\times SB'+c\times SC'+\ldots l\times SL' = SZ'(a+b+c+d+\ldots l).$$

Ducatur per S recta quævis SN'', in quam demittantur perpendicula AA'', BB'', CC'', $DD''\ldots LL''$, ZZ''; dico, fore etiam $a\times AA''+b\times BB''+c\times CC''+\ldots l\times LL''=ZZ''(a+b+c+\ldots+l)$.

Etenim quam facillime demonſtratur (ſive immediate, ſive ex primis trigonometriæ planæ principiis de ſinu & coſinu ſummæ ac differentiæ duorum angulorum) eſſe

$$\begin{aligned}
AA'' &= AA'\,\text{coſ.}\,NSN''-SA'\,\text{ſin.}\,NSN''\\
BB'' &= BB'\,\text{coſ.}\,NSN''-SB'\,\text{ſin.}\,NSN''\\
CC'' &= CC'\,\text{coſ.}\,NSN''-SC'\,\text{ſin.}\,NSN''\\
&\vdots\\
LL'' &= LL'\,\text{coſ.}\,NSN''-SL'\,\text{ſin.}\,NSN''\\
ZZ'' &= ZZ'\,\text{coſ.}\,NSN''-SZ'\,\text{ſin.}\,NSN''.
\end{aligned}$$

Proinde

$$a\times AA''+b\times BB''c\times CC''+\ldots+l\times LL''=\begin{matrix}\text{coſ.}\,NSN''(a\times AA'+b\times BB'+c\times CC'+\ldots+l\times LL')\\ -\text{ſin.}\,NSN''(a\times SA'+b\times SB'+c\times SC+\ldots+l\times SL')\end{matrix}$$
$$=\begin{matrix}ZZ'(a+b+c+\ldots l)\,\text{coſ.}\,NSN''\\ -SZ'(a+b+c+\ldots l)\,\text{ſin.}\,NSN''\end{matrix}=(a+b+c+\ldots l)(ZZ'\,\text{coſ.}\,NSN''-SZ'\,\text{ſin.}\,NSN'')$$
$$=(a+b+c+\ldots l)\,ZZ''.$$

3°. Viciſſim: ſi propoſitio locum habet pro duabus rectis SN, SN'', eadem locum habet pro tertia recta SN' uni priorum perpendiculari & per S ducta.

Propoſitione igitur pro duabus quibuscunque rectis ſibi mutuo occurrenti-

 bus

bus ſtabilita, eadem vera eſt pro alia quacunque recta; ſive transeat per punctum occurſus priorum, ſive non.

Obſervatio. In enunciatis præcedentibus ſuppoſui, puncta data jacere ad easdem partes rectæ, in quam perpendicula aguntur. Si ſecus fuerit: mutatis ſignis perpendiculorum ex punctis ad diverſas ejusdem rectæ partes ſitis demiſſorum; quæcunque de ſumma dicta fuerunt, applicantur exceſſui, quo ſumma rectangulorum, punctis ab una hujus lineæ parte ſitis reſpondentium, ſuperat ſummam reliquorum. Quo monito præmiſſo, in ſequentibus pariter de ſumma tantum dicere ſufficiet; quaſi omnia puncta ad easdem rectæ, in quam perpendicula aguntur, partes jacerent.

§. 126. In Mechanica oſtenditur: punctum Z commune eſſe *gravitatis centrum* totidem maſſarum a, b, c, . . . l, quarum ſingularum centra gravitatis ſint A, B, C, . . . L. Qua puncti Z denominatione hic etiam (brevitatis cauſa) uti liceat.

Itaque punctum rectæ magnitudine datæ medium eſt centrum gravitatis hujus lineæ; & ſuperficies, quam recta magnitudine data circa axem quemlibet revoluta generat, æqualis eſt rectangulo ſub hac recta & ſub circumferentia, quam centrum gravitatis ejus rotatione hac deſcribit.

Item centrum figuræ cujusvis rectanguli ſimul eſt ejus centrum gravitatis; & ſolida rotatione rectangulorum circa axes quoscunque, uni ex lateribus eorum parallelos, genita ſunt in ratione compoſita ex rationibus ipſorum rectangulorum atque circumferentiarum, quas earum centra gravitatis deſcribunt.

Sint quotlibet rectæ in eodem plano poſitione ac magnitudine datæ, quæ ſimul circa axem quemlibet revolvantur. In punctis rectarum harum mediis fingantur maſſæ ſingulis rectis proportionales, & quæratur centrum gravitatis commune harum maſſarum. Summa ſuperficierum a rectis illis rotatione hac genitarum æqualis eſt rectangulo facto ex ſumma harum rectarum, & ex circumferentia ab communi maſſarum harum centro gravitatis eadem revolutione percurſa.

Sint quotlibet rectangula in eodem plano poſitione & magnitudine data; ſic ut ſingulorum horum rectangulorum unum latus ſit rectæ alicui poſitione datæ parallelum. Rectangula hæc ſimul revolvantur circa axem rectæ huic paral-

parallelum. Tum in centris figuræ horum rectangulorum concipiantur maſſæ iisdem proportionales, & quæratur centrum gravitatis commune omnium harum maſſarum; ſolidum ab rectangulis his revolutione illa genitum eſt in ratione compoſita ex ſumma rectangulorum, & ex circumferentia a communi gravitatis centro revolutione hac genita.

§. 187. *Lemma notum.* Trapezium $AA'B'B$, cujus anguli B, B' ſunt recti, rotetur circa latus $A'B'$; ſolidum revolutione hac genitum eſt ad cylindrum ejusdem altitudinis $A'B'$, & cujus radius baſeos eſt r, uti $\frac{AA'^2+AA'\times BB'+BB'^2}{3}$ ad r^2. Proinde, poſita π dimidia circumferentia circuli, cujus radius eſt unitas, capacitatis ſolidi hujus expreſſio eſt $A'B'\times\frac{AA'^2+AA'\times BB'+BB'^2}{3}\pi$. Fig. 32.

Theorema. Rectangulum quodcunque circa axem quemcunque extra rectangulum in plano ejus ſitum revolvatur: dico, ſolidum rotatione hac genitum eſſe in ratione compoſita ex magnitudine rectanguli & ex circumferentia a centro ejus rotatione hac genita.

Sit $ABCD$ rectangulum, cujus centrum S, quod revolvatur circa axem $A'D'$ extra illud in plano ipſius ſitum; ex S demittatur in hunc axem perpendiculum SS': dico, ſolidum rotatione rectanguli $ABCD$ circa axem $A'D'$ genitum eſſe in ratione compoſita ex rectangulo $ABCD$, & ex circumferentia, cujus radius eſt SS'; ſeu expreſſionem capacitatis hujus ſolidi eſſe $ABCD\times SS'\times 2\pi$. Fig. 35.

Sint a, b, c, d puncta media laterum AB, BC, CD, DA.
Ex omnibus punctis A, B, C, D, a, b, c, d, demittantur in axem $A'D'$ perpendiculares AA', BB', CC', DD', aa', bb', cc', dd'. Sit φ angulus, ſub quo SS' inclinatur ad latera oppoſita rectanguli AB, CD.

Solidum rotatione rectanguli genitum eſt exceſſus, quo ſolidum rotatione ſpatii $A'ABCC'$ genitum ſuperat ſolidum genitum rotatione ſpatii $A'ADCC'$; ſeu ſol.$ABCD$ = ſol.$A'ABB'$ — ſol.$D'DCC'$ + ſol.$B'BCC'$ — ſol.$A'ADD'$

$$= \tfrac{1}{3}\pi.A'B'\left\{\begin{array}{l} A'A^2+A'A\times BB'+BB'^2 \\ -(D'D^2+D'D\times CC'+CC'^2)\end{array}\right\}$$

$$+ \tfrac{1}{3}\pi.A'D'\left\{\begin{array}{l} B'B^2+B'B\times CC'+CC'^2 \\ -(A'A^2+A'A\times DD'+DD'^2)\end{array}\right\}$$

 $= \tfrac{1}{3}\pi.$

$$=\tfrac{1}{3}\pi.A'B'\left\{(BB'^2-DD'^2)-(CC'^2-AA'^2)+\frac{(BB'+AA')^2-(BB'-AA')^2}{4}-\frac{(CC'+DD')^2-(CC'-DD')^2}{4}\right\}$$

$$=\tfrac{1}{3}\pi.A'D'\left\{(BB'^2-DD'^2)+(CC'^2-AA'^2)+\frac{(BB'+CC')^2-(BB'-CC')^2}{4}-\frac{(AA'+DD')^2-(AA'-DD')^2}{4}\right\}$$

$$=\tfrac{1}{3}\pi.A'B'\{2SS'(BB'-DD')-2SS'(CC'-AA')+SS'(BB'+AA'-CC'-DD')\}$$

$$+\tfrac{1}{3}\pi.A'D'\{2SS'(BB'-DD')+2SS'(CC'-AA')+SS'(BB'-AA'+CC'-DD')\}$$

$$\begin{matrix}=\pi.A'B'.SS'(\;AA'+BB'-CC'-DD')\\ +\pi.A'D'.SS'(-AA'+BB'+CC'-DD')\end{matrix} = \begin{matrix}2\pi.A'B'.SS'(aa'-cc')\\ +2\pi.A'D'.SS'(bb'-dd')\end{matrix}$$

Atqui $(aa'-cc') = BC\,\text{ſin.}\,\varphi$ Item $A'B' = AB\,\text{ſin.}\varphi$

$(bb'-cc') = AB\text{coſ.}\varphi$ $A'D' = BC\,\text{coſ.}\varphi$

Itaque $$\begin{matrix}2\pi.A'B'.SS'(aa'-cc')\\ +2\pi.A'D'.SS'(bb'-dd')\end{matrix} = 2\pi.SS'\times \begin{matrix}AB\cdot BC\,\text{ſin.}^2\varphi\\ +AB\times BC\text{coſ.}^2\varphi\end{matrix} = 2\pi.SS'.AB\cdot BC =$$

$$= ABCD\cdot SS'.2\pi.$$

Corollarium primum. Rectangula quotcunque circa eundem axem revolvantur: ſumma ſolidorum, quæ rotatione rectangulorum horum generantur, æqualis eſt prismati, cujus baſis eſt ſumma horum rectangulorum, & cujus altitudo eſt circumferentia percurſa a centro gravitatis communi totidem maſſarum in centris rectangulorum coadunatarum ſingulisque rectangulis proportionalium; quod punctum dicitur centrum gravitatis commune omnium horum rectangulorum.

Corollarium ſecundum. Sint quotcunque rectangula poſitione & magnitudine data, quæ circa axem quempiam revolvuntur. Solidum rotatione hac genitum proportionale eſt diſtantiæ centri gravitatis communis omnium rectangulorum ab axe rotationis.

Scholium. Modo haud multum abſimili demonſtratur: ſolidum a parallelogrammo obliquangulo circa axem quemcunque revoluto genitum æquale eſſe prismati, cujus baſis eſt ipſum parallelogrammum, & cujus altitudo æqualis eſt circumferentiæ a centro figuræ parallelogrammi rotatione hac genitæ. Et ſimilia inde nectuntur corollaria.

§. 128.

§. 128. Solidum, factum ex aliqua figura atque ex diſtantia centri gravitatis figuræ ab aliqua recta, vocatur *momentum figuræ* reſpectu hujus rectæ. Proinde momentum alicujus rectanguli reſpectu axis, circa quem rotatur, eſt ad ſolidum ab eodem rectangulo hac rotatione genitum in ratione conſtanti; nempe in ratione radii ad circumferentiam, ſeu ut $1 : 2\pi$. Et momentum quotlibet rectangulorum reſpectu axis, circa quem rotantur, eſt ad ſummam ſpatiorum ſolidorum ab iisdem rectangulis hac rotatione genitorum, in eadem ratione conſtanti.

Pariter rectangulum, factum ex aliqua linea atque ex diſtantia centri ejus gravitatis a recta quapiam, dicitur *momentum* prioris *lineæ* reſpectu poſterioris; & rectangulum, factum ex ſumma quotcunque linearum & ex diſtantia communis earum centri gravitatis ab aliqua recta, vocatur momentum harum linearum reſpectu hujus rectæ. Proinde ſi rectæ quotlibet circa axem quempiam revolvantur, momentum harum rectarum reſpectu hujus axis eſt ad ſummam ſuperficierum, quas eædem lineæ hac rotatione generant, in prædicta ratione conſtanti, $1 : 2\pi$.

His præmiſſis pergo ad ſuperficiem, a linea quacunque circa axem aliquem rotata genitam, & ad ſolidum rotatione cujuscunque figuræ circa axem aliquem genitum.

§. 129. Brevitatis cauſa denotent $S.R$, $S.F$ ſolida a rectangulis figuræ alicui inſcriptis aut circumſcriptis, & a figura ipſa rotatione circa eundem axem ſimul genita; porro denotent $M.R$, $M.F$ momenta horum extenſorum reſpectu hujus axis; $G.R$, $G.F$ diſtantias centrorum gravitatis horum extenſorum ab eodem axe; & $A.R$, $A.F$ areas horum extenſorum.

1°. Eſt
$$\begin{aligned} M.F(=A.F\times G.F) &= \text{lim.}\, M.R \\ &= \text{lim.}(A.R\times G.R) \\ &= \text{lim.}A.R\times \text{lim.}G.R \\ &= A.F\times \text{lim.}G.R \\ \text{proinde } G.F &= \text{lim.}G.R. \end{aligned}$$

2°. Eſt

2°. Eſt $S.F = \lim. S.R$

$$= 2\pi.\lim. M.R$$

$$= 2\pi.\lim.(A.R \times G.R)$$

$$= 2\pi.\lim. A.R \times \lim. G.R$$

$$= 2\pi. A.F \times G.F$$

$$= 2\pi \times M.F.$$

Proinde *ſolidum rotatione figuræ F genitum æquale eſt ſolido facto ex hac figura atque ex circumferentia a centro gravitatis ejus rotatione hac genita; ſeu ſolidum a figura F genitum eſt ad momentum ejus reſpectu axis rotationis in ratione conſtanti circumferentiæ circuli ad radium; & ſolida rotatione ejusdem figuræ circa varios axes genita ſunt inter ſe uti diſtantiæ centri gravitatis hujus figuræ ab iisdem axibus.*

Eodem modo *ſuperficies rotatione alicujus lineæ circa axem quempiam genita æqualis eſt rectangulo, facto ſub hac linea & ſub circumferentia a centro gravitatis ejus rotatione hac genita; ſeu ſuperficies hæc eſt ad momentum lineæ genitricis reſpectu axis, circa quem rotatur, in ratione conſtanti circumferentiæ circuli ad radium.* (a)

§. 130.

(a) Theorema hoc, ſuppreſſa demonſtratione, Pappus jam ſub finem *Præfationis Lib. VII. Collect. math.* expoſuit his verbis: *Ο των τελειων αμφοισικων λογος συνηπται εκ τε των αμφοισματων και των επι τυς αξονας ομοιως κατηγμενων ευθειων απο των εν αυτοις κεντροβαρικων σημειων. Ο δε των ατελων εκ τε των αμφοισματων και των περιφερειων, οσας εποιησε τα εν αυτοις κεντροβαρικα σημεια.* Quæ Halleius (*Apollonii De ſectione rationis & ſpatii Libri.* Oxon. 1706.) ita vertit: „Figuræ perfecto gyro genitæ rationem habent compoſitam ex ratione gyrantium & ex illa rectarum ſimiliter ad axes ductarum ab ipſarum gyrantium gravitatis centris. Ratio „vero incompleto gyro genitarum fit ex ratione gyrantium & arcuum, quos deſcripſere earundem centra gravitatis.„ Commandinus (Pappi *mathemat. Collect.* Piſauri 1588.) verterat: „Perfectorum utrorumque ordinum proportio compoſita eſt „ex proportione amphismatum & rectarum linearum ſimiliter ad axes ductarum a punctis, quæ in ipſis gravitatis centra ſunt. Imperfectorum autem proportio compoſita eſt ex proportione amphismatum & circumferentiarum a punctis, quæ in ipſis „ſunt centra gravitatis, factarum.„ Keplerus (*Nova Stereometria doliorum.* Lincii 1615. P. I. Theor. XVIII. ſq.) omnem annulum, genitum rotatione cujuslibet figuræ ſymmetricæ circa axem diametro figuræ parallelum, ſive extra figuram ſitum, ſive perimetrum ejus contingentem, æqualem eſſe oſtendit cylindro, cujus altitudo æquet longitudinem circumferentiæ, quam centrum figuræ circumductæ deſcripſit,

baſis

§. 130. Hinc deducuntur methodi centrum gravitatis cujusvis figuræ determinandi. Nempe determinetur ſolidum rotatione figuræ hujus circa axem quemlibet genitum; quod transformetur in priſma, cujus baſis ſit ipſa figura rotata: altitudo hujus prismatis imminuta in ratione circumferentiæ ad radium erit diſtantia centri gravitatis hujus figuræ ab axe rotationis. Et proinde centrum gravitatis ipſum determinabitur, ſi determinentur ſolida rotatione figuræ hujus circa duos axes (ſibi invicem non parallelos) genita.

Viciſſim, centro gravitatis figuræ alicujus poſitione dato, determinantur ſolida rotatione figuræ hujus circa axem quemlibet poſitione datum genita.

Eadem applicantur ad ſuperficies rotatione alicujus lineæ circa duos axes genitas.

Quodſi autem figura ſymmetrica eſt, ſeu axem aliquem figuræ habet: quoniam centrum gravitatis ejus in hoc axe ſitum eſt, ut poſitio centri hujus determinetur, ſufficit momentum figuræ quærere, reſpectu unius rectæ, v. gr. reſpectu lineæ, quæ ſit axi huic ordinatim applicata.

Caput decimum quintum.

De ſolidis earumque ſuperficiebus in genere, et de curvis duplicis curvaturæ.

Solidorum rotundorum proprietates tum a figura genitrice, tum a poſitione axis, circa quem figura hæc rotatur, pendere, in tribus capitibus præcedentibus abunde fuit declaratum. Solida hæc præcipue mathematicos occuparunt. Dantur autem innumera alia a ſolidis rotundis diverſa; quorum potiores duas ſpecies

baſis vero eadem ſit cum ſectione annuli. Guldinus (*De centro gravitatis*. Lib. II. Viennæ 1640. Cap. VIII. Prop. II.) regulam, quam vocat, generalem compoſitionis poteſtatum rotundarum hanc tradidit, ſed per inductionem tantum comprobavit: „Quantitas rotunda in viam rotationis (lineam circularem, quam in rotatione deſcribit centrum gravitatis magnitudinis rotatæ) ducta producit poteſtatem rotundam „uno gradu altiorem poteſtate ſive quantitate rotata.„ Demonſtrationes regulæ, quæ ad Guldinum tanquam inventorem referri conſuevit, varii deinde varias propoſuerunt. Conf. Montucla *Hiſt. des Math.* T. II. p. 19. ſqq.

ſpecies primum breviter pertractare, tum generatim de illis disquirere non alienum ab re erit.

§. 131. Sit figura quæcunque plana poſitione & magnitudine data. Recta quæpiam ita moveatur, ut ſibi ſemper parallela maneat, & ut aliquod ejus punctum circa perimetrum figuræ hujus progrediatur. Singula rectæ hujus puncta deſcribunt lineas perimetro figuræ datæ parallelas, eidemque ſimiles & æquales: proinde ſolidi hoc modo geniti ſectio, plano quocunque figuræ datæ parallelo facta, æqualis & ſimilis eſt huic figuræ. Superficies a recta mobili genita dicitur *ſuperficies cylindrica;* & recta mobilis dicitur ejus *latus.* Figura data, & ſectio ſolidi, plano figuræ huic parallelo & per alterum lateris extremum transeunte facta, dicuntur *baſes* ſolidi. Solidum ipſum, baſibus & ſuperficie cylindrica terminatum, dicitur *cylindrus* ſeu ſolidum cylindricum. Cylindrus eſt *rectus* aut *obliquus*, prouti latus eſt baſi perpendiculare aut obliquum. Diſtantia baſium cylindri dicitur ejus *altitudo.*

Solidum cylindricum limes eſt prismatum ipſi circumſcriptorum aut inſcriptorum. Sed capacitates horum prismatum ſunt in ratione compoſita ex rationibus baſium & altitudinum eorum; & baſes ſolidi cylindrici ſunt etiam limites baſium horum prismatum: proinde etiam ſolida cylindrica ſunt in ratione compoſita ex rationibus baſium & altitudinum ipſorum. Ideo denotante S capacitatem ſolidi cylindrici, B baſin ejus, & H altitudinem; erit $S = BH$.

Proinde quotiescunque area baſis ſolidi cylindrici terminis finitis exprimi poteſt; capacitas quoque ſolidi cylindrici accurate determinatur.

Pariter ſuperficies cylindricæ rectæ ſunt in ratione compoſita ex rationibus perimetrorum baſium, & altitudinum. Superficies autem cylindricæ obliquæ pendent a perimetro ſectionis cylindri, plano ipſius lateribus perpendiculari factæ.

§. 132. Solida inter, quæ a cylindricis originem ducunt, ea contemplari ſufficiat, quæ cylindrorum rectorum (inprimis) ſectione baſi obliqua generantur.

Fig. 36. Sit nempe AB recta quæcunque in plano baſis cylindri recti acta; & ſit $ANN'B$ ſegmentum baſis recta hac AB abſciſſum. Cylindrus ſecetur plano per

AB

AB tranſeunte; ſitque $AMM'B$ ſectio ſuperficiei cylindricæ plano hoc facta. Solidum, ſuperficie cylindrica $ANN'BM'M$, & ſegmentis $ANN'B$, $AMM'B$ comprehenſum, vocatur *cono-cuneus* vel *ungula cylindrica.*

In plano baſis ducatur recta quævis NP ipſi AB perpendicularis; per NP agatur planum baſi perpendiculare; & ſit MNP ſectio ungulæ plano hoc facta. Trianguli rectanguli MNP angulus MPN æqualis eſt angulo inclinationis duorum planorum $ANN'B$, $AMM'B$, qui ſit φ; & proinde omnes ſectiones pari modo factæ dantur ſpecie; nempe eſt $NM = NP$ tang. φ, $N'M' = N'P'$ tang. φ.

Capacitas ungulæ cylindricæ limes eſt capacitatis ungularum prismaticarum, quæ oriuntur ex ſectionibus ſimul factis prismatum ſolido cylindrico inſcriptorum aut circumſcriptorum. Capacitate igitur ungulæ poſita $=S$, axis AB abſciſſa $AP = x$, & $NP = y$; fit $\frac{dS}{dx} = \frac{1}{2}yy$ tang. φ: proinde ungulæ capacitas ab integratione hujus formulæ pendet.

Exempla. Solidum cylindricum ſit cylindrus circularis rectus; & recta AB ſit diameter $2r$ baſis cylindri. Erit $yy = 2rx - xx$; $\frac{dS}{dx} = \frac{1}{2}(2rx - xx)$ tang. φ; $S = C + \frac{1}{2}(rxx - \frac{1}{3}x^3)$ tang. φ. Atqui $S = 0$, quando $x = 0$; proinde $S = \frac{1}{2}xx(r - \frac{1}{3}x)$ tang. φ. Sit $x = 2r$: erit $S = 2rr(\frac{1}{3}r)$ tang. $\varphi = \frac{2}{3}rr \cdot r$ tang. φ. Cubatura ſolidi hujus accurate obtinetur, neque a quadratura circuli pendet. Par ratio eſt ungularum cylindricarum, quarum baſes ſunt ellipſes conicæ, & AB alteruter earum axis, v. gr. transverſus $2a$. Tum ſcilicet $yy = \frac{bb}{aa}(2ax - xx)$; unde $S = \frac{1}{2}\frac{bb}{aa}$ tang. $\varphi \cdot xx(a - \frac{1}{3}x)$. Sit $x = 2a$; erit $S = 2bb$ tang. $\varphi \cdot \frac{1}{3}a = \frac{2}{3}ab \cdot b$ tang. φ.

Idem dicatur de ungulis ex ſectionibus cylindrorum parabolicorum & hyperbolicorum genitis.

Obſervatio. Ex formula $\frac{dS}{dx} = \frac{1}{2}yy$ tang. φ ſequitur: ungulas, ſectionibus ejusdem ſolidi per eandem in baſi rectam tranſeuntibus factas, inter ſe eſſe uti tangentes angulorum φ. Verum hæc ratio tantum ſubſiſtit, quamdiu tang. φ eſt poſſibilis, ſeu quamdiu planum $AMM'B$ cylindri lateribus occurrit. Quando autem anguli φ tangens fit impoſſibilis, ſeu quando angulus φ fit $90°$; ſigni $\frac{1}{0}$

 ſeu

ſeu ∞ introductione monemur: non amplius de ungulis agi poſſe; nec formulam pro ipſis traditam poſſe ad capacitatem ſolidi, plano baſi normali a cylindro abſciſſi, determinandam applicari.

§. 133. Æquatio differentialis ſuperficiei curvæ ungularum ex iisdem principiis deducitur. Sit nempe z arcus AN ſegmenti baſis $ANN'B$, & ſit S ſuperficies curva ungulæ; ſit $\frac{dS}{dz} = MN = y \operatorname{tang.} \varphi$: unde res ad calculum integralem reducitur.

Exemplum primum. Solidum cylindricum ſit cylindrus circularis rectus, & ſit $AB = 2r$ diameter baſis; erit $\frac{dz}{dx} = \frac{r}{y}$: hinc $\frac{dS}{dx} = r \operatorname{tang.} \varphi$; $S = rx \operatorname{tang.} \varphi$. Sit $x = 2r$; erit $S = 2rr \operatorname{tang.} \varphi = 2r \times r \operatorname{tang.} \varphi$: proinde hoc caſu etiam ſuperficies curva ungulæ abſolute habetur.

Scholium. Eadem, quæ §. præcedente, de impoſſibilitate applicationis harum formularum ad caſum, quo $\varphi = 90°$, obſerventur.

Exemplum ſecundum. Solidum cylindricum ſit cylindrus ellipticus, & ſit AB axis alteruter ellipſis.

1°. Sit $AB = 2a$ axis transverſus, $2b$ axis ſecundus, ſitque $aa - bb = ee$; & abſciſſæ x axis AB ſumantur a centro: erit $\frac{dS}{dx} = \frac{be}{aa} \operatorname{tang.} \varphi \sqrt{\left(\frac{a^4}{ee} - xx\right)}$; cujus formulæ integratio non ab rectificatione ellipſis, ſed a quadratura tantum circuli pendet. Nempe eſt $S = \frac{1}{2}\left(\frac{a^3}{e} \operatorname{arc.ſin.} \frac{ex}{aa} + \frac{ex}{a}\sqrt{\left(\frac{a^4}{ee} - xx\right)}\right) \operatorname{tang.} \varphi$.

Sit $x = a$; ſit $S = \frac{1}{2}\left(\frac{a^3}{e} \operatorname{arc.ſin.} \frac{e}{a} + ab\right) \operatorname{tang.} \varphi$, & ſuperficies integra $AMM'BN'N$ eſt $\left(\frac{a^3}{e} \operatorname{arc.ſin.} \frac{e}{a} + ab\right) \operatorname{tang.} \varphi$.

Scholium. $\operatorname{Arc.ſin.} \frac{e}{a} = \frac{e}{b} - \frac{1}{3}\frac{e^3}{b^3} + \frac{1}{5}\frac{e^5}{b^5} - \ldots\ldots$ (§. 79.) hinc $\frac{a^3}{e} \operatorname{arc.ſin.} \frac{e}{a} = \frac{a^3}{b}\left(1 - \frac{1}{3}\frac{ee}{bb} + \frac{1}{5}\frac{e^4}{b^4} - \ldots\right)$: proinde caſu, quo $e = 0$, ſeu quo baſis eſt circulus, ſit $S = (aa + ab) \operatorname{tang.} \varphi = 2aa \operatorname{tang.} \varphi$ (ut prius).

2°. Sit AB axis ſecundus $= 2b$: erit $\frac{dS}{dx} = \frac{ae}{bb} \operatorname{tang.} \varphi \sqrt{\left(\frac{b^4}{ee} + xx\right)}$; cujus formulæ integratio non a rectificatione ellipſis, ſed ab quadratura hyperbolæ ſeu a logarithmis pendet. Nempe eſt

$S =$

$$S = \tfrac{1}{2}\frac{ae}{bb}\text{tang.}\,\varphi\Big(x\sqrt{\Big(\frac{b^4}{ee}+xx\Big)} + \frac{b^4}{ee}\log.\frac{\sqrt{\big(\frac{b^4}{ee}+xx\big)}+x}{\frac{bb}{e}}\Big].$$

Sit x = b; ſuperficies integra eſt $\frac{ae}{bb}\text{tang.}\,\varphi\Big(\frac{bba}{e}+\frac{b^4}{ee}\log.\frac{\frac{bba}{e}+b}{\frac{bb}{e}}\Big] =$

$$= \Big(aa+\frac{bba}{e}\log.\frac{a+e}{b}\Big)\text{tang.}\,\varphi = \Big(aa+\frac{bba}{e}\log.\sqrt{\frac{a+e}{a-e}}\Big)\text{tang.}\,\varphi =$$

$$= \Big(aa+\frac{bba}{e}\log.\sqrt{\frac{1+\frac{e}{a}}{1-\frac{e}{a}}}\Big]\,\text{tang.}\,\varphi.$$

Scholium. $\text{Log.}\sqrt{\frac{1+\frac{e}{a}}{1-\frac{e}{a}}} = \frac{e}{a}+\frac{1}{3}\frac{e^3}{a^3}+\frac{1}{5}\frac{e^5}{a^5}+\ldots.$ (§. 61.);

hinc $\frac{bba}{e}\log.\sqrt{\frac{a+e}{a-e}} = bb\big(1+\frac{1}{3}\frac{ee}{aa}+\frac{1}{5}\frac{e^4}{a^4}+\ldots.\big)$

Unde, facto $e=0$, ſuperficies propoſita eſt $2aa\,\text{tang.}\,\varphi$ (ut prius).

Eodem modo oſtenditur: ſuperficiem ungularum parabolicarum aut abſolute haberi, aut ad logarithmos reduci; & ſuperficies ungularum hyperbolicarum (ſectione per alterutrum axem genitarum) a rectificatione hyperbolæ non pendere.

§. 134. Solida conica aliud conſtituunt ſolidorum genus, quorum conſideratio frequenter occurrit.

Sit figura quæcunque plana poſitione & magnitudine data. Sit etiam punctum quodvis extra planum figuræ poſitione datum; & recta per hoc punctum ducta circa perimetrum figuræ rotetur. Recta hæc ita revoluta gignit *ſuperficiem conicam*, cujus *vertex* eſt punctum datum. Recta ex vertice ad punctum aliquod perimetri baſis ducta coincidit cum linea genitrice ea poſitione, qua per punctum hoc perimetri baſis tranſit; ideoque recta hæc tota in ſuperficie conica jacet, & dicitur *latus* ſuperficiei conicæ. Solidum, figura data & ſuper-

ficie conica terminatum, dicitur *conus*, ſeu ſolidum conicum; & figura ipſa ejus *baſis*. Recta ex vertice in planum baſis perpendiculariter demiſſa vocatur *altitudo* coni. Si baſis habet centrum figuræ, & altitudo coni plano baſis in hoc centro occurrit; conus vocatur *rectus*: ſi ſecus; conus eſt *obliquus*.

Si conus plano ſecetur baſi parallelo: figura ſectionis baſi ſimilis eſt; & dimenſiones homologæ ſectionis ac baſis ſunt inter ſe uti diſtantiæ plani ſecantis & baſis a vertice coni.

Sint duo coni æquealti; baſes eorum in eodem plano jaceant, ita ut ipſi ſint verſus easdem hujus plani partes ſiti. Altitudo communis duorum conorum dividatur in partes quotcunque æquales: & utrique inſcribantur ac circumſcribantur priſmata æquealta ad normam § primi exempli tertii; quorum baſes ſint figuræ ſimiles ſectionibus ſimul inſcriptæ aut circumſcriptæ. Duo priſmata in utroque cono ſibi invicem reſpondentia (ſeu a vertice coni æqualiter diſtantia) ſunt in ratione conſtanti baſium ſuarum, ſeu in ratione figurarum baſibus conorum ſimul inſcriptarum aut circumſcriptarum; proinde & ſummæ horum priſmatum ſunt in eadem ratione conſtanti. Quare & limites harum ſummarum, nempe duo coni, ſunt uti limites figurarum baſibus inſcriptarum & circumſcriptarum, ſeu ut ipſæ baſes. Ratio igitur duorum conorum æquealtorum æqualis eſt rationi baſium eorum.

Atqui ratio baſium æqualis eſt rationi cylindrorum æquealtorum baſibus illis inſiſtentium. Proinde coni æquealti inter ſe ſunt uti cylindri æquealti iisdem baſibus inſiſtentes.

Sed conus circularis eſt pars tertia cylindri circularis æquealti ſuper eadem baſi. Proinde conus quilibet pars eſt tertia cylindri æquealti eidemque baſi inſiſtentis.

Determinatio igitur capacitatis cujusvis ſolidi conici reducitur ad determinationem baſis.

§. 135. Secus autem res habet quod ad ſuperficiem ſolidorum conicorum. Notum eſt, ſuperficiem coni recti circularis a circumferentia circuli pendere. Determinatio autem ſuperficiei coni circularis obliqui (cujus inveſtigationi

tioni tot tantique mathematici incubuerunt) nequidem ad rectificationem ſectionum conicarum terminis finitis poteſt reduci.

Omiſſis caſibus particularibus, quibus compendia quædam poſſunt indagationi ſuperficierum conicarum adhiberi, methodum omnium univerſaliſſimam disquiſitionem hanc inſtituendi ſtrictim exponam.

Sit *AMB* baſis coni; *S* vertex ejus; *SP* ipſius altitudo, quæ plano baſis in *P* occurrit. Plano baſis circumſcribatur figura quæcunque rectilinea; per ſingula figuræ baſi circumſcriptæ latera & per verticem coni agantur plana; quæ conſtituent ſuperficiem pyramidis cono circumſcriptæ. Sit *M* punctum contactus unius ex lateribus; in quod ex vertice *S* demittatur perpendiculum *ST*. Sit *c* latus figuræ baſin in *M* contingens. Ducantur *SM*, *PM* rectæ. Superficies trianguli, cujus hoc latus eſt baſis, & cujus vertex *S*, eſt $\frac{1}{2}c \times ST = \frac{1}{2}c \times SM \text{ſin.} SMT$. In angulo ſolido *M*, cujus acies ſunt *MP*, *MS*, *MT*, duæ facies *SMP*, *PMT* ſibi invicem ſunt perpendiculares: proinde $\text{coſ.} SMT = \text{coſ.} PMT \, \text{coſ.} SMP$; & $\text{ſin.} SMT = \surd(1 - \text{coſ.}^2 PMT \, \text{coſ.}^2 SMP)$. Superficies conica ponatur $= S$, & arcus $AM = z$: erit

Fig. 37.

$$\frac{dS}{dz} = \tfrac{1}{2} SM \surd(1 - \text{coſ.}^2 PMT \, \text{coſ.}^2 SMP)$$
$$= \tfrac{1}{2} \surd(SM^2 - PM^2 \text{coſ.}^2 PMT)$$
$$= \tfrac{1}{2} \surd(SP^2 + PM^2 \, \text{ſin.}^2 PMT).$$

Atqui ex æquatione data baſis datur angulus *PMT* per *PM* & per angulum *APM* v. gr.; daturque etiam exponens differentialis $\frac{dz}{d.PM}$: proinde res ad calculum integralem ſemper reducitur.

Exemplum. Baſis ſit circulus, cujus centrum *C*, & radius *CM*; & punctum *P* jaceat intra circulum. Erit $\text{ſin.} PMT = \text{coſ.} CMP$; $PM \, \text{ſin.} PMT = PM \, \text{coſ.} CMP = CM - CP \, \text{coſ.} MCP$. Sit $CM = r$, $SP = h$, $CP = a$, $MCP = z$: erit $\frac{dS}{dz} = \frac{1}{2} r \surd(hh + rr - 2ar \, \text{coſ.}\, z + aa \, \text{coſ.}^2 z)$. Sit $hh + rr = bb$; $2r = c$; fit $\frac{dS}{dz} = \frac{1}{2} r \surd(bb - a \, \text{coſ.}\, z(c - a \, \text{coſ.}\, z)) = \frac{1}{2} br \surd\left(1 - \frac{a}{b} \text{coſ.}\, z\left(\frac{c - a \, \text{coſ.}\, z}{b}\right)\right)$

$= \frac{1}{2} br$

$$= \tfrac{1}{2} br \left\{ 1 - \frac{\frac{1}{2}}{1} \frac{a \operatorname{coſ.} z (c - a \operatorname{coſ.} z)}{bb} \right.$$

$$- \frac{\frac{1}{2}}{1} \cdot \frac{\frac{1}{2}}{2} \frac{aa \operatorname{coſ.}^2 z (c - a \operatorname{coſ.} z)^2}{b^4}$$

$$- \frac{\frac{1}{2}}{1} \cdot \frac{\frac{1}{2}}{2} \cdot \frac{\frac{3}{2}}{3} \frac{a^3 \operatorname{coſ.}^3 z (c - a \operatorname{coſ.} z)^3}{b^6}$$

$$- \frac{\frac{1}{2}}{1} \cdots \frac{\frac{5}{2}}{4} \frac{a^4 \operatorname{coſ.}^4 z (c - a \operatorname{coſ.} z)^4}{b^8}$$

$$- \frac{\frac{1}{2}}{1} \cdots \frac{\frac{7}{2}}{5} \frac{a^5 \operatorname{coſ.}^5 z (c - a \operatorname{coſ.} z)^5}{b^{10}}$$

$$\left. - \quad - \quad - \quad - \quad - \quad - \right\}$$

Unde, ſumtis integralibus arcui $z = 180^\circ$ reſpondentibus, ſuperficies integra coni obliqui, fit

$$S = br\pi \Big(1 + \frac{\frac{1}{2}}{1} \cdot \frac{1}{2} \frac{aa}{bb}$$

$$- \frac{\frac{1}{2}}{1} \cdot \frac{\frac{1}{2}}{2} \frac{\frac{1}{2} aacc + \frac{\frac{4}{1} \cdot \frac{3}{2}}{2^4} a^4}{b^4}$$

$$+ \frac{\frac{1}{2}}{1} \cdot \frac{\frac{1}{2}}{2} \cdot \frac{\frac{3}{2}}{3} \frac{3 \cdot \frac{\frac{4}{1} \cdot \frac{3}{2}}{2^4} a^4 cc + \frac{\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3}}{2^6} a^6}{b^6}$$

$$- \frac{\frac{1}{2}}{1} \cdots \frac{\frac{5}{2}}{4} \frac{\frac{\frac{4}{1} \cdot \frac{3}{2}}{2^4} a^4 c^4 + \frac{4}{1} \cdot \frac{3}{2} \frac{\frac{6}{1} \cdot \frac{5}{2} \cdot \frac{4}{3}}{2^6} a^6 c^2 + \frac{\frac{8}{1} \cdots \frac{5}{4}}{2^8} a^8}{b^8}$$

$$+ \frac{\frac{1}{2}}{1} \cdots \frac{\frac{7}{2}}{5} \frac{5 \cdot \frac{\frac{6}{1} \cdots \frac{4}{3}}{2^6} a^6 c^4 + \frac{5}{1} \cdot \frac{4}{2} \cdot \frac{\frac{8}{1} \cdots \frac{5}{4}}{2^8} a^8 c^2 + \frac{\frac{10}{1} \cdots \frac{6}{5}}{2^{10}} a^{10}}{b^{10}}$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

$$+ \quad - \quad - \quad - \quad - \quad - \quad - \quad - \Big)$$

Series hæc eo promtius convergit, quo b ſeu $\sqrt{(hh+rr)}$ major eſt reſpectu a & c, ſeu a & $2r$. Ea nititur hoc principio: quod, ſi fuerit $\frac{dZ}{dz} = \operatorname{coſ.}^{2m} z$, eſt $Z = \frac{\frac{2m}{1} \cdot \frac{2m-1}{2} \cdots \frac{m+1}{m}}{2^{2m}} \pi$, integrali ſumto pro $z = 180^\circ$.

Eodem

Eodem modo ſeries obtinetur admodum regularis pro $z = 90^\circ$, ex integratione formularum $\frac{dZ}{dz} = \text{coſ.}^{2m+1} z$, facto $z = 90^\circ$.

His (introductionis loco) de ſolidis cylindricis & conicis breviter præmiſſis, pergo ad conſiderationem ſolidorum magis univerſalem.

§. 136. Quemadmodum punctorum in eodem plano jacentium ſitus mutui omnium frequentiſſime determinantur, puncta hæc ad duas rectas ſibi invicem normales referendo per rectas ad eas perpendiculariter actas; & quemadmodum curvarum in eodem plano jacentium ſymptomata felici ſucceſſu determinantur per relationem mutuam perpendiculorum, ex ſingulis curvarum harum punctis in duas rectas ſibi invicem normales demiſſorum: ita etiam ſitus mutui punctorum in ſpatio ſitorum ſæpiſſime determinantur, puncta hæc ad tria plana ſibi invicem normalia referendo per rectas ad hæc plana perpendiculariter actas; pariterque tam curvarum in eodem plano non jacentium, quam ſuperficierum curvarum, & ſolidorum ſuperficiebus his terminatorum ſymptomata frequentiſſime determinantur per relationem mutuam rectarum, ex ſingulis curvarum harum aut ſuperficierum punctis in tria plana poſitione data & ſibi invicem normalia perpendiculariter demiſſarum.

Sint $\begin{array}{c}SX\\ SY\\ SZ\end{array}$ ſectiones communes planorum $\begin{array}{cc}YSX & XSZ\\ XSY, & YSZ\\ XSZ & ZSY\end{array}$ ſibi invicem normalium & in puncto S ſibi mutuo occurrentium. Sit M punctum aliquod in ſpatio, ex quo agatur recta SP plano XSZ perpendicularis. Tum ex puncto P agatur recta PQ rectæ SX perpendicularis. Rectæ PQ, SQ reſpective æquales ſunt rectis ex eodem puncto M in plana YSX, YSZ perpendiculariter actis. Proinde puncti M ſitus in ſpatio determinatur tribus rectis Fig. 38.

MP, PQ, SQ

in plana ZSX, YSX, YSZ perpendiculariter actis, quæ vocentur

y, z, x.

Hinc v. gr. determinantur tam diſtantia SM puncti M a vertice S, quam inclinationes rectæ hujus SM ad tria plana propoſita: fit enim

$$SM^2 = MP^2 + PS^2 = MP^2 + PQ^2 + SQ^2 = yy + zz + xx.$$

 tang.

$$\text{tang.} MSP = \frac{MP}{SP} = \frac{y}{\sqrt{(zz+xx)}}$$

$$\text{ſin.} MSP = \frac{MP}{SM} = \frac{y}{\sqrt{(yy+zz+xx)}}$$

$$\text{coſ.} MSP = \frac{SP}{SM} = \sqrt{\left(\frac{zz+xx}{yy+zz+xx}\right)}.$$

§. 137. Tria plana, *ZSX*, *YSX*, *YSZ* nonniſi majoris facilitatis gratia ſumuntur ſibi invicem perpendicularia; quæcunque enim de hoc ſitu illorum planorum dicuntur, facile poſſunt ad alium quemcunque ſitum transferri.

Fig. 58. 1°. Per *S* agatur quodvis aliud planum *R*, quod plano *ZSX* in recta *SV*
2°. occurrat ſub angulo dato α; & angulus *XSV* vocetur β. Ex *P* puncto agatur *PR* ipſi *SV* perpendicularis; & planum per rectas *MP*, *PR* ductum occurrat plano propoſito in *Rr* recta: in quam ex puncto *M* agatur *MT* perpendicularis, quæ proinde erit plano propoſito *TRV* perpendicularis; & angulus *PRr* eſt angulus α, ſub quo *rRV* planum inclinatur ad planum *ZSX*.

Erit (§. 125.)

$$\begin{aligned}
PR &= PQ\,\text{coſ.}\beta + SQ\,\text{ſin.}\beta \\
SR &= SQ\,\text{coſ.}\beta - PQ\,\text{ſin.}\beta \\
MT (&= PR\,\text{ſin.}\alpha - MP\,\text{coſ.}\alpha) \\
&= PQ\,\text{coſ.}\beta\,\text{ſin.}\alpha + SQ\,\text{ſin.}\beta\,\text{ſin.}\alpha - MP\,\text{coſ.}\alpha \\
RT (&= PR\,\text{coſ.}\alpha + MP\,\text{ſin.}\alpha) \\
&= PQ\,\text{coſ.}\beta\,\text{coſ.}\alpha + SQ\,\text{ſin.}\beta\,\text{coſ.}\alpha + MP\,\text{ſin.}\alpha.
\end{aligned}$$

Proinde tres rectæ *SR*, *RT*, *MT*, quibus puncti *M* ſitus reſpectu plani *R* determinatur, habentur per angulos α, β, & per rectas *MP*, *PQ*, *SQ* expreſſæ.

2°. Planum propoſitum R' non transeat per *S*; ſed rectæ *SX* occurrat in S', & plano *ZSX* in $S'V'$. Per *S* agatur recta *SV* ipſi $S'V'$ parallela, & per *SV* agatur planum *R* plano R' parallelum. Tum iisdem, quæ in caſu præcedente factis, occurrat *PR* ipſi $S'V'$ in R', & *MT* occurrat in T' plano R'; ac proinde ſit $R'T'$ ipſi *RT* parallela. Erit

$RR' = SS'\,\text{ſin.}\beta$, unde $PR' = PQ\,\text{coſ.}\beta + SQ\,\text{ſin.}\beta + SS'\,\text{ſin.}\beta$

$S'R' = SR + SS'\,\text{coſ.}\beta$, ſeu $S'R' = SQ\,\text{coſ.}\beta - PQ\,\text{ſin.}\beta + SS'\,\text{coſ.}\beta$

$R'T' = RT + R'R\,\text{coſ.}\alpha$, ſeu $R'T' = PQ\,\text{coſ.}\beta\,\text{coſ.}\alpha + SQ\,\text{ſin.}\beta\,\text{coſ.}\alpha + MP\,\text{ſin.}\alpha + SS'\,\text{ſin.}\beta\,\text{coſ.}\alpha$

$MT' = MT + R'R\,\text{ſin.}\alpha$, ſeu $MT' = PQ\,\text{coſ.}\beta\,\text{ſin.}\alpha + SQ\,\text{ſin.}\beta\,\text{ſin.}\alpha - MP\,\text{coſ.}\alpha + SS'\,\text{ſin.}\beta\,\text{ſin.}\alpha.$

Viciſ-

Viciſſim rectæ MP, PQ, SQ exprimi poſſunt per rectas $S'R'$, $R'T'$, MT', ſimul cum recta SS' & angulis α & β. Fit enim

$$SQ = S'R'\text{coſ.}\beta + R'T'\text{ſin.}\beta\,\text{coſ.}\alpha + MT'\text{ſin.}\beta\,\text{ſin.}\alpha - SS'$$

$$PQ = -S'R'\text{ſin.}\beta + R'T'\text{coſ.}\beta\,\text{coſ.}\alpha + MT'\text{coſ.}\beta\,\text{ſin.}\alpha$$

$$MP = RT'\text{ſin.}\alpha - MT'\text{coſ.}\alpha.$$

Corollarium primum. Sit $MT'=0$; ſeu planum R' per M transeat. Erit

$$MP = MR'\text{ſin.}\alpha$$

$$PQ = -S'R'\,\text{ſin.}\beta + R'M\text{coſ.}\beta\,\text{coſ.}\alpha$$

$$SQ = S'R'\,\text{coſ.}\beta + R'M\text{ſin.}\beta\,\text{coſ.}\alpha - SS'.$$

Corollarium ſecundum. Obſervandum eſt: rectas $S'R'$, $R'T'$, MT' per rectas MP, PQ, SQ ita determinari, ut in expreſſionibus ipſarum rectæ MP, PQ, SQ nulla alia operatione invicem connectantur præter additionem aut ſubſtractionem, ſeu priorum rectarum expreſſiones eſſe primi tantum gradus functiones poſteriorum.

§. 138. Sit M' aliud punctum, ex quo agatur $M'P'$ recta plano ZSX perpendicularis; & ex puncto P' agatur $P'Q'$ recta ipſi SX perpendicularis. Tum agatur etiam PP' recta: ac ſint Pq' ipſi $P'Q'$, Mm' ipſi $M'P'$ perpendiculares. Fig. 38. 1°.

Sint $M'P'$, $P'Q'$, SQ',
y', z', x' reſpective.

$$MM'^2 = Mm'^2 + M'm'^2 = PP'^2 + M'm'^2 = Pq'^2 + P'q'^2 + M'm'^2$$
$$= (x'-x)^2 + (z'-z)^2 + y'-y)^2$$

unde
$$MM' = \sqrt{((x'-x)^2 + (z'-z)^2 + (y'-y)^2)}$$

Angulus $M'Mm'$ ille eſt, quo MM' recta ad planum ZSX inclinatur; atque

$$\text{tang.}M'Mm = \frac{M'm'}{PP'} = \frac{y'-y}{\sqrt{((x''-x)^2 + (z'-z)^2)}}$$

$$\text{ſin.}M'Mm = \frac{y'-y}{\sqrt{((x'-x)^2 + (z'-z)^2 + (y'-y)^2)}}$$

$$\text{coſ.}M'Mm = \sqrt{\left(\frac{(x'-x)^2 + (z'-z)^2}{(y'-y)^2 + (x'-x)^2 + (z'-z)^2}\right)}.$$

Obſervatio. Recta PP' projectio eſt orthographica rectæ MM' in planum

 ZSX;

ZSX; & ratio rectæ MM' ad projectionem ſuam PP' æqualis eſt rationi radii ad coſinum anguli, quo recta MM' ad planum *ZSX* inclinatur.

Univerſim notum eſt: figuræ planæ cujuslibet in planum aliquod orthographice projectæ rationem ad projectionem ſuam æqualem eſſe rationi radii ad coſinum anguli, quo figuræ hujus planum ad planum projectionis inclinatur.

§. 139. Porro ſit & tertium punctum M''. Sit pariter $M''P''$ plano *ZSX*, & $P''Q''$ rectæ *SX* perpendicularis. Agantur MM'', $M'M''$ rectæ; & ſint Pq'', Mm'' rectis $P''Q''$, $M''P''$ perpendiculares. Sint $\underset{y''}{M''P''}$, $\underset{z''}{P''Q''}$, $\underset{x''}{SQ''}$ reſpective.

Erit $MM'' = \sqrt{((x''-x)^2+(y''-y)^2+(z''-z)^2)}$

$M'M'' = \sqrt{((x''-x')^2+(y''-y')^2+(z''-z')^2)}$.

Hinc determinatur area trianguli $MM'M''$ per formulam notam

$$MM'M'' = \sqrt{\left(\frac{MM'+MM''+M'M''}{2}\cdot\frac{MM'+M'M''-M'M''}{2}\cdot\frac{MM'-MM''+M'M''}{2}\cdot\frac{-MM'+MM''+M'M''}{2}\right)}$$

$$= \tfrac{1}{4}\sqrt{\left\{\begin{array}{l} 2MM'^2.MM''^2 - MM'^4 \\ +2MM'^2.M'M''^2 - MM^4 \\ +2MM''^2.M'M''^2 - M'M''^4 \end{array}\right\}}.$$

Item $PP' = \sqrt{((x'-x)^2+(z'-z)^2)}$

$PP'' = \sqrt{((x''-x)^2+(z''-z)^2)}$

$P'P'' = \sqrt{((x''-x')^2+(z''-z')^2)}$: unde pariter determinatur area trianguli $PP'P''$.

Hinc etiam determinatur capacitas trunci prismatici triangularis $MM'M''P''P'P$, cujus expreſſio eſt $PP'P'' \times \frac{MP+M'P'+M''P''}{3}$.

Hinc quoque infertur angulus, ſub quo planum trianguli $MM'M''$ ad planum *ZSX* inclinatur; coſinus enim hujus anguli eſt $\frac{PP'P''}{MM'M''}$. Ad ſequentia autem præſtat, angulum hunc paulo aliter inveſtigare, ut ſequitur.

§. 140. Rectæ $M'M$, $M''M$ productæ, rectis $P'P$, $P''P$ in punctis T', T'' occurrant. Rectæ PT', PT'' dantur magnitudine; ſit nempe

$$PT' = MP \times \frac{PP'}{M'm'} = y \times \frac{\sqrt{((x'-x)^2+(z'-z)^2)}}{y'-y}$$

$$PT'' = MP \times \frac{PP''}{M''m''} = y \times \frac{\sqrt{((x''-x)^2+(z''-z)^2)}}{y''-y}.$$

In

In triangulo $T'PT''$ dantur angulus $T'PT''(=p'pp'')$, & crura hujus anguli PT', PT''; proinde datur recta $T'T'' = \sqrt{(T'p^2 - 2T'p.PT''\text{coſ}.T'PT'' + PT''^2)}$. Sit pT ipſi $T'T''$ perpendicularis; erit $pT = \frac{T'p.PT''\text{ſin}.T'PT''}{T'T''}$
$= \frac{T'p.PT''\text{ſin}.T'PT''}{\sqrt{(T'p^2 - 2T'p.PT''\text{coſ}.T'PT'' + PT''^2)}}$.
Hinc tang. $MTP(=\frac{Mp}{Tp}) = \frac{Mp\sqrt{(T'p^2 - 2T'p.PT''\text{coſ}.T'PT'' + PT''^2)}}{T'p.PT''\text{ſin}.T'PT''}$.

Exemplum. Angulus $p'pp''$ ſit rectus: fit tang. $MTP =$ $\sqrt{\left(\frac{(y''-y)^2}{(x''-x)^2+(z''-z)^2} + \frac{(y'-y)^2}{(x'-x)^2+(z'-z)^2}\right)}$. Sint autem ſimul plana $MPP'M'$, $MPP''M''$, planis YSX, YSZ reſpective parallela; & proinde $x''=x$, $z'=z$: erit tang. $MTP = \sqrt{\left(\left(\frac{y''-y}{z''-z}\right)^2 + \left(\frac{y'-y}{x'-x}\right)^2\right)}$. Porro angulus TPQ ille eſt, ſub quo SX, $T'T''$ rectæ ad ſe invicem inclinantur; angulus hic fit æqualis angulo T', cujus tangens eſt $\frac{PT''}{PT'} = \frac{z''-z}{y''-y} : \frac{x'-x}{y'-y}$. Fig. 38. 3°.

§. 141. Hiſce præmiſſis de punctis in ſpatio, quorum ſitus mutui nulla relatione determinata ſecum invicem connectuntur: pergo ad præcipuum disquiſitionis hujus caput, ad caſum nempe, quo puncta relatione aliqua data inter ſe connectuntur; ita ut locus punctorum M ſit ſuperficies aut curva aliqua, relatione quadam inter perpendicula MP, PQ, SQ determinata. Quem ut, quantum potero, luculenter explicem, ab exemplis omnium ſimpliciſſimis ordiar.

Exemplum primum. Detur ſumma quadratorum perpendiculorum MP, PQ, SQ; ſumma hæc eſt MS^2 (§. 136.): proinde recta SM datur magnitudine, & puncta M jacent in ſuperficie ſphærica, cujus centrum eſt S, & cujus radius magnitudine datur. Fig. 38. 1°. 3°.

Quoniam ſumma $MP^2 + PQ^2 + SQ^2$ datur magnitudine: ſi recta MP ejusdem manet magnitudinis, & proinde puncta M in plano poſitione dato & plano ZSX parallelo jacent; ſumma $PQ^2 + SQ^2$ ſeu SP^2 pariter datur magnitudine. Agatur MN rectæ SP parallela, quæ ipſi SY in N occurrat. Recta $NM = SP$

etiam datur magnitudine; & proinde puncta M ſita ſunt in circumferentia circuli, cujus centrum N & radius $NM = SP$.

Hinc diſcimus: omnes ſolidi propoſiti ſectiones planis ZSX, YSX, YSZ parallelis factas eſſe circulares, & centra harum ſectionum jacere in recta perpendiculari plano dato, cui plana acta ſunt parallela; ideoque ſolidum propoſitum eſt ſolidum rotationis circa quamlibet rectarum SX, SY, SZ.

Fig. 38. 2°. Secetur idem ſolidum alio quocunque plano per rectam $S'V'$ tranſeunte. Sit angulus, ſub quo hoc planum ad planum ZSX inclinatur $= \alpha$, & angulus $V'S'X = \beta$; ſitque M punctum aliquod perimetri ſectionis.

Quoniam (§. 137.)

$$MP = MR'\,\text{ſin.}\alpha$$
$$PQ = MR'\,\text{coſ.}\alpha\,\text{coſ.}\beta - S'R'\,\text{ſin.}\beta$$
$$SQ = MR'\,\text{coſ.}\alpha\,\text{ſin.}\beta + S'R'\,\text{coſ.}\beta - S'S$$

$$PQ^2 + SQ^2 = MR'^2\,\text{coſ.}^2\alpha - 2SS' \times MR'\,\text{coſ.}\alpha\,\text{ſin.}\beta + S'R'^2 - 2SS' \times S'R'\,\text{coſ.}\beta + S'S^2$$
$$\& \; MP^2 + PQ^2 + SQ^2 = MR'^2 - 2SS' \times MR'\,\text{coſ.}\alpha\,\text{ſin.}\beta + S'R'^2 - 2SS' \times S'R'\,\text{coſ.}\beta + S'S^2$$
$$= (MR' - S'S\,\text{coſ.}\alpha\,\text{ſin.}\beta)^2 + (S'R' - SS'\,\text{coſ.}\beta)^2 + SS'^2\,\text{ſin.}^2\alpha\,\text{ſin.}^2\beta,$$

quæ eſt æquatio circumferentiæ circuli; proinde ſectio ſolidi plano quolibet facta eſt circulus (uti notum).

Scholium. Ex formula $MP^2 + PQ^2 + SQ^2 - SS'^2\,\text{ſin.}^2\alpha\,\text{ſin.}^2\beta = \begin{matrix}(MR' - S'S\,\text{coſ.}\alpha\,\text{ſin.}\beta)^2 \\ + (S'R' - S'S\,\text{coſ.}\beta)^2\end{matrix}$, datis duabus trium quantitatum SS', α, β, tertia ſic determinatur, ut planum ductum tangat (ſi fieri poſſit) ſolidum propoſitum; facto nempe $SS'\,\text{ſin.}\alpha\,\text{ſin.}\beta = \sqrt{(MP^2 + PQ^2 + SQ^2)}$: ſed de hoc ſitu planorum actorum ſeorſim agere præſtat.

Exemplum ſecundum. Summa quadratorum rectarum PQ, SQ habeat rationem datam ad quadratum reliquæ MP.

Fig. 38. 1°. & 3°. I. Magnitudine rectæ MP manente eadem; ſumma quadratorum rectarum PQ, SQ, ſeu quadratum rectæ SP, pariter eſt datæ magnitudinis. Proinde ſectiones ſolidi propoſiti, planis plano ZSX parallelis factæ, ſunt circulares. Quoniam ratio $MP : SP$ ſeu $SN : NM$ datur: in triangulo MSN rectangulo ad N angulus MSN datur magnitudine; ideoque SM eſt latus coni recti, cujus axis eſt SY. Sit φ angulus datus YSM.

II. Recta

II. Recta SQ eadem manente, quadratum rectæ MP eſt ad quadratum rectæ PQ ſpatio data auctum, in ratione data; proinde omnes ſectiones planis plano ZSY parallelis factæ ſunt hyperbolæ ſimiles. Ejuſdemque indolis ſunt ſectiones planis plano YSX parallelis factæ.

III. Solidum propoſitum plano ſecetur quocunque, quod plano ZSX in $S'V'$ Fig. 38. 2°.
recta occurrat ſub angulo α, & ſit angulus $XS'V'=\beta$.

$$\text{Eſt } PQ^2+SQ^2=SR^2 = MR'^2\text{coſ.}^2\alpha+S'R'^2 \begin{matrix}-2SS'\times MR'\text{coſ.}\alpha\,\text{ſin.}\beta \\ -2SS'\times S'R'\text{coſ.}\beta\end{matrix} +SS'^2$$

$$\text{et } MP^2 = MR'^2\text{ſin.}^2\alpha;$$

$$\text{proinde } MR'^2\text{ſin.}^2\alpha : MR'^2\text{coſ.}^2\alpha+S'R'^2-2SS'\times \begin{matrix}MR'\text{coſ.}\alpha\,\text{ſin.}\beta \\ +S'R'\text{coſ.}\beta\end{matrix} +SS'^2 = 1:\text{tang.}^2\varphi:$$

$$\text{unde } MR'^2\text{coſ.}^2\alpha : (SR'-SS'\text{coſ.}\beta)^2+(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2 = \text{cot.}^2\alpha:\text{tang.}^2\varphi.$$

1°. Sit $\text{cot.}\alpha=\text{tang.}\varphi$: erit $(SR'-SS'\text{coſ.}\beta)^2=MR'^2\text{coſ.}^2\alpha-(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2$ $=(2MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)\times SS'\text{ſin.}\beta$; proinde ſectio ſolidi eſt parabolica.

2°. Sit $\text{cot.}\alpha \gtrless \text{tang.}\varphi$;

$$\begin{aligned}(SR'-SS'\text{coſ.}\beta)^2 &= MR'^2\text{ſin.}^2\alpha\,\text{tang.}^2\varphi-(MR'\text{coſ.}\alpha-SS'\text{ſin.}\beta)^2 \\ &=(MR'(\text{ſin.}\alpha\,\text{tang.}\varphi+\text{coſ.}\alpha)-SS'\text{ſin.}\beta)(SS'\text{ſin.}\beta-MR'(\text{coſ.}\alpha-\text{ſin.}\alpha\,\text{tang.}\varphi)) \\ &=\left(MR'\frac{\text{coſ.}\overline{\varphi-\alpha}}{\text{coſ.}\varphi}-SS'\text{ſin.}\beta\right)\left(SS'\text{ſin.}\beta-MR'\frac{\text{coſ.}\overline{\varphi+\alpha}}{\text{coſ.}\varphi}\right) \\ &=\frac{\text{coſ.}\overline{\varphi-\alpha}\,\text{coſ.}\overline{\varphi+\alpha}}{\text{coſ.}^2\varphi}\left(SS'^2\frac{\text{ſin.}^2\alpha\,\text{ſin.}^2\beta\,\text{coſ.}^2\varphi}{\text{coſ.}\overline{\varphi-\alpha}\,\text{coſ.}\overline{\varphi+\alpha}}-\left(MR'-SS'\frac{\text{ſin.}\beta\,\text{coſ.}\alpha\,\text{coſ.}^2\varphi}{\text{coſ.}\overline{\varphi-\alpha}\,\text{coſ.}\overline{\varphi+\alpha}}\right)^2\right).\end{aligned}$$

Quare ſectio eſt ellipſis, ſi fuerit $\text{coſ.}\overline{\varphi-\alpha}\,\text{coſ.}\overline{\varphi+\alpha}$ quantitas poſitiva, ſeu $\varphi+\alpha<180°$; hyperbola autem, ſi fuerit $\varphi+\alpha>180°$.

Et quoniam ſpecies ſectionis pendet ab coëfficiente $\text{coſ.}\overline{\varphi-\alpha}\,\text{coſ.}\overline{\varphi+\alpha}$, omnes ſectiones planis ſibi invicem parallelis factæ ſunt inter ſe ſimiles.

Exemplum tertium. Summa MP^2+PQ^2 ſemper ſit rectangulo ſub recta NS Fig. 38. 1°. & 3°.
& ſub recta aliqua data $2p$ æqualis.

Igitur $SP^2=MN^2=2p\times SN$; proinde ſolidum propoſitum gignitur rotatione parabolæ, cujus vertex S, parameter $2p$, & axis SY.

Secetur ſolidum plano ipſi YSZ parallelo: recta SQ manente eadem, erit $PQ^2=2p\times MP-SQ^2$; proinde ſectio eſt parabolica: idemque valet de ſectionibus plano YSX parallelis.

Sece-

Secetur autem ſolidum plano per $S'V'$ transeunte. Erit

Fig. 38. 2°. $(S'R' - SS'\text{coſ.}\beta)^2 + (MR'\text{coſ.}\alpha - SS'\text{ſin.}\beta)^2 = 2p \times MR'\text{ſin.}\alpha$.

1°. Sit $\alpha = 90°$; ideoque coſ.$\alpha = 0$: erit $(SR' - SS'\text{coſ.}\beta)^2 - (SS'\text{ſin.}\beta)^2 = 2p \times MR'$. Proinde omnes ſectiones planis plano ZSX perpendicularibus factæ ſunt parabolæ, quarum parameter $2p$ eſt eadem quæ parabolæ genitricis.

2°. Sit $\alpha \gtrless 90°$. Fit $(S'R' - SS'\text{coſ.}\beta)^2 =$
$\text{coſ.}^2\alpha\,(p\,\text{tang.}\alpha\,\text{ſec.}^2\alpha\,(p\,\text{tang.}\alpha + 2SS'\text{ſin.}\beta) - (MR' - \text{ſec.}\alpha\,(p\,\text{tang.}\alpha + SS'\text{ſin.}\beta))^2$.

Sectio igitur eſt elliptica, & ſpecies ellipſeos pendet ab angulo α. Proinde omnes ſectiones planis ſibi invicem parallelis factæ ſunt inter ſe ſimiles.

Ut ſectio poſſibilis ſit, debet eſſe
$MR' - \text{ſec.}\alpha(p\text{tang.}\alpha + SS'\text{ſin.}\beta) \overline{\gtrless}\, \text{ſec.}\alpha\sqrt{(p\text{tang.}\alpha(p\text{tang.}\alpha + 2SS'\text{ſin.}\beta))}$; & ſi fuerit $MR' = \text{ſec.}\alpha\,(p\text{tang.}\alpha + SS'\text{ſin.}\beta) + \text{ſec.}\alpha\sqrt{(p\text{tang.}\alpha(p\text{tang.}\alpha + 2SS'\text{ſin.}\beta))}$, planum ductum tangit ſuperficiem propoſitam.

Pauca hæc exempla ſufficiunt ad dijudicandum, quomodo ex relatione data perpendiculorum MP, PQ, SQ ſymptomata ſolidi poſſunt deduci; & nominatim, quomodo ex æquatione hac proprietates ſectionum ejus, planis poſitione datis factarum, inferuntur. Et quoniam æquatio curvæ, quæ eſt ſectio ſolidi propoſiti per aliquod planum, oritur ex æquatione data inter tres perpendiculares MP, PQ, SQ, ſubſtituendo valores harum linearum per rectas MR', SR' expreſſos; neque hæ in valoribus iſtis alia operatione præter additionem & ſubſtractionem afficiuntur (§. 137.): gradus æquationis pro ſectione ortæ ſuperare non poteſt gradum æquationis, quæ relationem trium perpendiculorum MP, PQ, SQ determinat.

§. 142. Transeo ad plana contactus ſolidorum generaliter conſideratorum. Situs plani cujuscunque determinatur per duas rectas ſibi mutuo vel parallelas vel occurrentes. Proinde ut determinetur planum, quod propoſitam ſuperficiem curvam in puncto dato contingat; determinandæ ſunt duæ rectæ ex hoc puncto ductæ, per quas planum hoc agi debeat. Secetur itaque ſolidum propoſitum duobus planis per punctum datum transeuntibus: tum determinatis ſectionum æquationibus (juxta §. 141.) ducantur (Cap. V.) rectæ, quæ ſectiones

ctiones genitas in puncto dato contingant. Planum per duas has tangentes transiens erit planum contactus.

Facillime autem propoſitum peragitur, ſi ſolidum ad tria plana *ZSX*, *YSX*, *YSZ* ſibi invicem perpendicularia referatur per rectas planis his perpendiculariter ordinatim applicatas; pariterque duæ ſectiones fiant planis duobus ex illis parallelis; proinde tertio perpendicularibus.

Sit igitur *M* punctum datum in ſuperficie ſolidi, ex quo in planum *ZSX* perpendicularis demittatur *MP*. Tum per rectam *MP* agantur duo plana *M'MP*, *M''MP*, planis *YSX*, *YSZ* reſpective parallela; & rectæ, quæ in puncto *M* ſectiones planis *M'MP*, *M''MP* factas tangunt, plano *ZSX* in *T'* & *T''* punctis occurrant. Erit (§. 40.) $PT' = y\frac{dx}{dy}$, $PT'' = y\frac{dz}{dy}$; hinc $T'T'' = y\sqrt{\left(\left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)}$. Fig. 38. 3°.

Sit *PT* ipſi *T'T''* perpendicularis; erit

$$PT = y \times \frac{\frac{dx}{dy}\cdot\frac{dz}{dy}}{\sqrt{\left(\left(\frac{dx}{dy}\right)^2 + \left(\frac{dz}{dy}\right)^2\right)}} = y \times \frac{1}{\sqrt{\left(\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}}$$

$$\text{tang.}\,MTP = \frac{MP}{PT} = \sqrt{\left(\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{(\text{tang.}^2 MT'P + \text{tang.}^2 MT''P)}.$$

$$\text{ſec.}\,MTP = \sqrt{\left(1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$\text{coſ.}\,MTP = \frac{1}{\sqrt{\left(1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)}}$$

$$\text{ſin.}\,MTP = \sqrt{\left[\frac{\left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2}{1 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dy}{dx}\right)^2}\right]}.$$

Exemplum primum. Curva in plano *M'MP* deſcripta ſit circulus, cujus radius *r*; & curva in plano *M''MP* ſit parabola, cujus parameter $2p$.

Itaque $yy = rr - zz$; $\frac{dy}{dz} = -\frac{z}{y}$; $\left(\frac{dy}{dz}\right)^2 = \frac{zz}{rr - zz}$

$yy = 2px$; $\frac{dy}{dx} = \frac{p}{y}$; $\left(\frac{dy}{dx}\right)^2 = \frac{p}{2x}$

$$\text{tang.}\,MTP = \sqrt{\left(\frac{zz}{rr - zz} + \frac{p}{2x}\right)}.$$

Exemplum ſecundum. Ambæ curvæ in planis $M'MP$, $M''MP$ ſint circuli, quorum radii r & r':

$$\text{erit} \quad \begin{matrix} \left(\frac{dy}{dz}\right)^2 = \frac{zz}{rr-zz} \\ \left(\frac{dy}{dx}\right)^2 = \frac{xx}{rr-xx} \end{matrix} ; \quad \text{tang.} MPT = \sqrt{\left(\frac{zz}{rr-zz} + \frac{xx}{rr-xx}\right)}.$$

§. 143. Ex præcedentibus deduci etiam poſſunt capacitates ſolidorum (ſive non ſint rotunda, ſive ut talia non conſiderentur), dummodo ſectiones eorum planis ſibi invicem parallelis factæ juxta datam legem creſcant aut decreſcant.

Etenim ſolidum propoſitum ſecetur planis ſibi invicem parallelis, & quorum diſtantiæ ſint v. gr. æquales. Tum ſolido inſcribantur & circumſcribantur ſolida cylindrica, quorum baſes ſint ſectiones ſolidi, & altitudo communis ſit diſtantia duarum ſectionum ſibi proximarum. Fruſtum ſolidi inter duas ſectiones vicinas comprehenſum, majus eſt ſolido cylindrico ipſi inſcripto; minus autem circumſcripto. Atqui ratio æqualitatis limes eſt rationis horum ſolidorum cylindricorum; ergo a fortiori ratio æqualitatis limes eſt rationis fruſti ſolidi ad alterutrum horum cylindrorum. Quare poſita diſtantia communi duarum ſectionum vicinarum $= \Delta x$, area ſectionis ſolidi $= s$, & capacitate ſolidi $= S$: erit $\lim.\frac{\Delta S}{dx} = s$; ideoque $\frac{dS}{dx} = s$. Data igitur lege, juxta quam ſectiones s creſcunt aut decreſcunt; datur exponens differentialis $\frac{dS}{dx}$; & proinde res ad calculum integralem reducitur.

Fig. 39. *Exemplum primum.* Sit $ASA'DD'$ paraboloides, gyratione dimidii ſegmenti parabolici SAC circa axem SC genita, quæ ſecetur plano MZM' baſi ADA' perpendiculari, & plano SDD' per axem transeunti parallelo. Sectio MZM' pariter erit ſegmentum parabolicum: nempe $ZP = SC \times \frac{AP.PA'}{AC^2} = SC \times \frac{MP^2}{AC^2}$. Igitur $ZMP(=\frac{2}{3}MP.ZP) = \frac{2}{3}SC.\frac{MP^3}{AC^2}$, & $ZMM' = \frac{4}{3}SC \times \frac{MP^3}{AC^2}$. Quare poſitis $CP = x$, $SC = b$, $AC = r$: erit $\frac{dS}{dx} = \frac{4}{3}b \times \frac{(rr-xx)^{\frac{3}{2}}}{rr}$. Unde

$S=b$

$S = b \times \frac{\frac{1}{3}x(5rr - 2xx)\sqrt{(rr-xx)} + \frac{1}{2}r^4 \text{arc.ſin.}\frac{x}{r}}{rr}$. Sit $x = r$: erit $S = \frac{1}{2}brrp$. Unde ſegmentum $MAZM' = b(\frac{1}{2}rr\,\text{arc.ſin.}\frac{\sqrt{(rr-xx)}}{r} - \frac{1}{3}x\sqrt{(rr-xx)}\frac{5rr-2xx}{2rr})$.

Exemplum ſecundum. Solidum $ASA'DD'$ ſit dimidia ellipſois, quadrante ellipſis SAC circa axem SC rotato genita. Sint ACS, DSD', ACD tria plana ſibi invicem perpendicularia; & ſit ZMM' planum plano DSD' parallelum.

Sint $SC = a$, $AC = b$, $CP = x$: erit $ZP = \frac{a}{b}.MP$; $MZP = \frac{1}{2}p \times MP \times ZP = \frac{1}{2}p \times \frac{a}{b}.MP^2$; $MZM' = p.\frac{a}{b}.MP^2 = p.\frac{a}{b}(bb - xx) = \frac{dS}{dx}$; unde $S = p.\frac{a}{b}(bbx - \frac{1}{3}x^3) = p.\frac{a}{b}x(\frac{2}{3}bb + \frac{1}{3}MP^2)$. Sit $x = b$; ſit $S = \frac{2}{3}abb \times p$. Hinc ſegmenti $MAZM'$ capacitas eſt $(\frac{2}{3}ab(b-x) - \frac{1}{3}(\frac{a}{b}x.MP^2)p$.

Exemplum tertium. Solidum propoſitum ſit conus rectus ASA', qui ſecetur plano parabolico MZM' lateri SA' parallelo. Fig. 40.

Hoc caſu eſt $\frac{dS}{dx} = MZM'\,\text{ſin.}\,ZPA$.

$MZM' = \frac{4}{3}.MP.ZP$; $\frac{dS}{dx} = \frac{4}{3}MP \times ZP\,\text{ſin.}ZPA = \frac{4}{3}(r-x)\sqrt{(rr-xx)}\,\text{ſin.}SAA'$;

$S = (\frac{2}{3}rx\sqrt{(rr-xx)} + \frac{2}{3}r^3\,\text{arc. ſin.}\frac{x}{r} - \frac{4}{9}(rr-xx)^{\frac{3}{2}})\text{ſin.}SAA' + C$.

Sit $S = 0$, quando $x = 0$; $C = \frac{4}{9}r^3\text{ſin.}SAA'$.

Hinc $S = \frac{4}{9}(AC^3 - MP^3)\text{ſin.}SAA' + \frac{2}{3}AC \times CP \times MP\,\text{ſin.}SAA' + \frac{2}{3}AC^2 \times MD\,\text{ſin.}SAA'$.

Scholium. Principiorum capite hoc explicatorum ad ſuperficies ſolidorum non rotundorum determinandas applicatio non æque facilis eſt; ob perpetuam mutationem angulorum, ſub quibus plana ſuperficiem tangentia ad plana ſecantia contigua inclinantur: neque magnam ad hunc ſcopum utilitatem formulæ inclinationum planorum tangentium ad plana, ad quæ ſuperficies refertur, §. 142. exhibita, afferre poſſe mihi videntur.

§. 144. Progredior ad curvas duplicis curvaturæ dictas, ſeu ad curvas in eodem plano non jacentes, quarum ſymptomata principiis capite hoc ſtabilitis determinantur. (a)

 Sit

(a) De his curvis primus egregium opuſculum ſcripſit ſagaciſſimus Clairaut, vix ſexdecim annos natus, ſub titulo: *Traité des Courbes à double Courbure*, Paris 1729.

Sit M punctum quodlibet alicujus curvæ duplicis curvaturæ, quæ referatur ad tria plana ZSX, YSX, YSZ, ſibi invicem normalia, per rectas MP, PQ, Fig. 38. SQ, perpendiculariter ſibi mutuo ordinatim applicatas. Curvæ hujus in planum ZSX projectio orthographica determinatur per relationem mutuam abſciſſarum SQ & ordinatim applicatarum PQ. Pariter ejusdem curvæ in planum YSZ projectio orthographica determinatur per relationem rectarum MP, PQ; denique curvæ hujus in planum YSX projectio orthographica determinatur per relationem rectarum MP, SQ. Hinc ſymptomatum curvæ duplicis curvaturæ determinatio revocatur ad conſiderationem curvarum in eodem plano jacentium, quæ in planis ZSX, YSZ, YSX ſunt projectiones orthographicæ curvæ propoſitæ.

Duabus autem harum projectionum cognitis innoteſcit tertia. Etenim data relatione perpendiculi MP ad unamquamque rectarum PQ, SQ (ſeu datis curvæ propoſitæ in plana YSX, YSZ projectionibus orthographicis); datur etiam (quantum permittit imperfecta functionum theoria) relatio rectarum PQ, SQ (ſeu projectio in planum ZSX). Pariterque data æquatione ſuperficiei curvæ, in qua curva duplicis curvaturæ jacet, & una trium curvæ in plana ZSX, YSX, YSZ projectionum; dantur duæ reliquæ. Data enim relatione mutua trium perpendiculorum MP, PQ, SQ per æquationem ſuperficiei propoſitæ determinata; dataque præterea relatione duarum v. gr. rectarum PQ, SQ per projectionem in planum ZSX: datur etiam relatio utriusque horum perpendiculorum ad tertium MP; ſeu dantur curvæ propoſitæ in plana YSZ, YSX projectiones.

Exemplum primum. Projectio in planum ZSX ſit circulus, cujus centrum S, radius r; erit ideo $zz+xx=rr$. Projectio autem in planum YSX ſit ellipſis, cujus centrum S, & cujus axes a, b in rectis SX, SY jaceant: erit itaque $yy=\frac{bb}{aa}(aa-xx)$, ſeu $xx=\frac{aa}{bb}(bb-yy)$;

$$\text{hinc } zz+xx=\frac{aa}{bb}(bb-yy)+zz=rr:$$

$$\text{unde } zz=\frac{aa}{bb}\left(yy+\frac{bb}{aa}(rr-aa)\right).$$ Curvæ igitur propoſitæ in planum YSZ projectio orthographica eſt hyperbolica.

Exem-

Exemplum ſecundum. Utraque projectio in plana ΥSX, ΥSZ ſint parabolæ conicæ, quarum axes jaceant in axe $S\Upsilon$, quarum parametri ſint $2p$, $2p'$; & quarum vextex communis ſit S.

Fit ideo $\begin{matrix} 2py = xx \\ 2p'y = zz \end{matrix}$; hinc $xx : zz = p : p'$, & $x : z = \sqrt{p} : \sqrt{p'}$: quare projectio in planum ZSX eſt linea recta; unde curva propoſita non eſt duplicis curvaturæ, ſed tota in eodem plano jacet per S transeunte, & plano ZSX perpendiculari.

Secus erit, ſi punctum S non ſit vertex communis utriusque parabolæ. Sit v. gr. $\begin{matrix} 2p(a+y) = xx \\ 2p'(b+y) = zz \end{matrix}$; hinc $\begin{matrix} 2py = xx - 2ap \\ 2p'y = zz - 2bp' \end{matrix}$:

$$\begin{aligned} \text{unde} \quad p'(xx - 2ap) &= p(zz - 2bp') \\ \text{ſeu} \quad xx &= \frac{p}{p'} zz - 2bp + 2ap \\ &= \frac{p}{p'}(zz - 2bp' + 2ap'); \end{aligned}$$

& proinde projectio in planum ZSX eſt hyperbola, niſi fuerit $b = a$.

Exemplum tertium. Curva propoſita jaceat in ſuperficie ſphærica, cujus centrum S, & radius datus $SM = R$. Curvæ autem in planum ZSX projectio orthographica ſit circumferentia circuli, cujus radius r, & centri C in plano ZSX poſitio determinetur per rectas $CA = a$, $Cb = b$, magnitudine datas. Fig. 38. 3°. & 4°.

Relatio trium perpendiculorum x, y, z determinatur duabus æquationibus $\begin{matrix} xx + yy + zz = RR \\ (b-x)^2 + (a-z)^2 = rr \end{matrix}$; hinc $a - z = \pm\sqrt{(rr - (b-x)^2)}$

$$zz = aa \pm 2a\sqrt{(rr - (b-x)^2)} + rr - (b-x)^2$$

$$xx + zz = aa \pm 2a\sqrt{(rr - (b-x)^2)} + rr - bb + 2bx = RR - yy;$$

unde $yy = RR - aa - rr + bb - 2bx \mp 2a\sqrt{(rr - (b-x)^2)}$.

Sit v. gr. $a = 0$, $b = 0$; ideoque centrum projectionis ſit in S, $yy = RR = rr$, quo docemur, projectiones in utroque plano ΥSX, ΥSZ eſſe rectas plano ZSX parallelas; & proinde hoc caſu lineam propoſitam eſſe ſimplicis curvaturæ.

Scholium. Exemplo hoc continetur inveſtigatio curvæ, quæ eſt ſectio communis ſuperficiei ſphæricæ, cujus centrum S & radius R; ac ſuperficiei cylindricæ plano ZSX perpendicularis, cujus centrum baſis eſt C & radius r. Pa-

 tetque

tetque hoc exemplo: quomodo inveſtigatio ſymptomatum ſectionis communis duarum ſuperficierum ad determinationem ſymptomatum curvarum duplicis curvaturæ reducatur, quod alio adhuc exemplo familiari illuſtrabo.

Exemplum quartum. Sit hemisphærium, cujus centrum S, planum baſis ZSX, & radius R: ſit porro conus rectus; cujus vertex C jaceat in plano baſis hemisphærii, ita ut baſis coni ſit baſi hemisphærii parallela, ſeu ut axis coni ſit baſi hemisphærii perpendicularis. Diſtantiæ verticis coni a planis YSX, YSZ dicantur a, b; & ſit φ dimidius angulus ſectionis coni plano per axem transeunte factæ. Erit

$$xx + zz + yy = RR$$

$$(b-x)^2 + (a-z)^2 = yy\,\text{tang.}^2\varphi$$

hinc $$xx\,\text{ſec.}^2\varphi - 2bx + bb + zz\,\text{ſec.}^2\varphi - 2az + aa = RR\,\text{tang.}^2\varphi$$

ſeu $$xx - 2bx\,\text{coſ.}^2\varphi + bb\,\text{coſ.}^2\varphi + zz - 2az\,\text{coſ.}^2\varphi + aa\,\text{coſ.}^2\varphi = RR\,\text{ſin.}^2\varphi$$

ſeu $$(x - b\,\text{coſ.}^2\varphi)^2 + (z - a\,\text{coſ.}^2\varphi)^2 = \text{coſ.}^2(RR - (aa+bb)\,\text{coſ.}^2\varphi).$$

Projectio igitur ſectionis communis ſuperficierum coni & ſphæræ in planum ZSX eſt circumferentia circuli.

Item fit $RR = aa \mp 2a\sqrt{(yy\,\text{tang.}^2\varphi - (b-x)^2)} + yy\,\text{ſec.}^2\varphi - bb + 2bx$

$RR = bb \mp 2b\sqrt{(yy\,\text{tang.}^2\varphi - (a-z)^2)} + yy\,\text{ſec.}^2\varphi - aa + 2az$, quæ ſunt æquationes projectionum ejusdem ſectionis in plana YSX, YSZ reſpective. (a)

Fig. 38. §. 145. Sint M, M' duo puncta curvæ alicujus curvaturæ, quorum projectiones in plano ZSX ſint P, P' reſpective; unde eſt PP' projectio chordæ MM', puncta M, M' jungentis, & $P'T'$ projectio ſecantis.

Quare $MM' = \sqrt{(Mm'^2 + M'm'^2)} = \sqrt{(PP'^2 + M'm'^2)} = \sqrt{(\Delta x^2 + \Delta z^2 + \Delta y^2)}$, et $\frac{MM'}{\Delta x} = \sqrt{\left(1 + \left(\frac{\Delta z}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$. Extendendo igitur ad curvas duplicis curvaturæ, quæ vera ſunt de curvis ſimplicis curvaturæ, quod ſcilicet ratio æqualitatis limes ſit rationis arcus ad chordam: $\frac{dS}{dx} = \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)}$; & proinde exponens differentialis arcus curvæ duplicis curvaturæ, & cujusvis perpen-

(a) Qui plura de hoc capite voluerit, conſulat Euleri *Introductionem ad Analyſin infinitorum: Appendix de Superficiebus.*

perpendiculorum ex puncto aliquo curvæ hujus in plana ZSX, ZSY, YSX demiſſorum, per projectiones curvæ hujus in hæc tria plana determinatur.

Porro anguli $MT'P$ limes eſt angulus, ſub quo tangens curvæ in M ad planum ZSX inclinatur, & tangens hujus anguli eſt $\frac{\frac{dy}{dx}}{\sqrt{\left(1+\left(\frac{dz}{dx}\right)^2\right)}}$. Item anguli $T'PQ$ ſeu $PP'q'$ limes eſt angulus, ſub quo recta PQ inclinatur ad planum per tangentem in M ductum ac plano ZSX perpendiculare; & cotangens hujus anguli eſt $\frac{dz}{dx}$.

§. 146. Quamvis methodus hic delineata proprietates curvarum duplicis curvaturæ inveſtigandi ſit omnium univerſaliſſima; occurrunt tamen caſus, quibus affectiones harum curvarum brevius & luculentius aliis modis eruuntur; perpendendo v. gr., quæ ex propoſita palmaria quadam proprietate harum curvarum conſequuntur.

Proponatur v. gr. curva in ſuperficie curva cylindri, quæ ad ſingula ejus latera ſub dato angulo inclinatur. Expandatur ſuperficies curva cylindri in planum; in plano hoc ducatur recta, quæ ſub dato angulo ad unum latus cylindri inclinatur: recta hæc erit expanſio curvæ propoſitæ.

Pariter deſcribenda ſit in ſuperficie curva coni circularis recti curva, quæ ad ſingula ejus latera ſub dato angulo inclinatur: ſuperficies hæc in planum expandatur; tum in plano hoc deſcribatur ſpiralis logarithmica, quæ ad radios ſectoris ſuperficiei cono æqualis ſub hoc angulo dato inclinatur. Spiralis hæc erit expanſio curvæ propoſitæ.

Idem potiſſimum illuſtratur exemplo *curvarum loxodromicarum* in ſuperficie ſphæræ deſcriptarum, quarum palmaria proprietas eſt, quod meridianos ſub eodem angulo dato interſecent. Sint P polus, PM, PM' meridiani ad duo puncta M, M' curvæ loxodromicæ ducti, MT tangens curvæ loxodromicæ in M, & $M'm$ arcus circuli paralleli centro P deſcripti. Sit $PMT = \varphi$. Sit PA meridianus poſitione datus, ad quem anguli APM referantur. Sit $MPM' = \Delta x$, $PM = y$, $Mm = -\Delta y$. Fig. 41.

lim.

$\lim.\frac{Mm}{M'm} = \cot.\varphi$, ſeu $\lim.\frac{-\Delta y}{\Delta x \,\text{ſin}.\, y} = \cot.\varphi$; $\lim.\frac{-\Delta y}{\Delta x} = \text{ſin}.\, y \cot.\varphi$.

$\lim. -\frac{\Delta x}{\Delta y} = \text{tang}.\,\varphi\, \text{coſec}.\, y$, ſeu $-\frac{dx}{dy} = \text{tang}.\,\varphi\,\text{coſec}.\, y$: unde $x = C + \text{tang}.\,\varphi \log.\cot.\tfrac{1}{2}y$. Sit $x = 0$, quando $y = 90° = p$: erit $C = 0$, & $x = \text{tang}.\,\varphi \log.\cot.\tfrac{1}{2}y$. Proinde cotangentibus dimidiorum arcuum *PM* in geometrica progreſſione creſcentibus, anguli *APM* creſcunt in progreſſione arithmetica.

Porro arcu *AM* poſito $= Z$, eſt $-\frac{dZ}{dy} = \text{ſec}.\,\varphi$; hinc $Z = C - y\,\text{ſec}.\,\varphi$. Sit $Z = 0$, quando $y = p$; $C = p\,\text{ſec}.\,\varphi$, $Z = MX\,\text{ſec}.\,\varphi$, cujus limes eſt $p\,\text{ſec}.\,\varphi$.

Hinc etiam determinatur area ſphærica *APM* inter radios vectores *PA*, *PM*, & arcum loxodromicum *AM* comprehenſa.

Etenim ſuperficie quadrantis hemisphærii dicta *H*, & area *APM* dicta *S*.

Fit $\frac{dS}{dx} = \frac{2H}{p}\text{ſin}.^2\tfrac{1}{2}y$;

atqui $\frac{dx}{dy} = -\text{tang}.\,\varphi\,\text{coſec}.\, y$;

proinde $\frac{dS}{dy} = -\frac{H}{p}\text{tang}.\,\varphi\,\text{tang}.\tfrac{1}{2}y$: unde $S = C - 2\frac{H}{p}\text{tang}.\,\varphi \log.\text{ſec}.\tfrac{1}{2}y$. Sit $S = 0$, quando $y = p$; $C = 2\frac{H}{p}\text{tang}.\,\varphi \log.\text{ſec}.45°$. $S = 2\frac{H}{p}\text{tang}.\,\varphi\left(\log.\frac{\text{ſec}.45°}{\text{ſec}.\frac{1}{2}y}\right)$, cujus limes eſt $2\frac{H}{p}\text{tang}.\,\varphi\log.\text{ſec}.45° = \frac{H}{p}\text{tang}.\,\varphi\log.2 = rr\,\text{tang}.\,\varphi \times \log.2 = \text{tang}.\,\varphi\log.2$ (radio ſphæræ pro unitate ſumto).

Scholium. Formulæ hæ iis tantum caſibus applicantur, quibus de loxodromica agitur, ſeu φ non eſt $= 90°$.

Caput decimum sextum.

De ſignificatione expreſſionis $\frac{0}{0}$ et æquipollentium.

§. 147.

Sit *P* quantitatis mutabilis *x* functio, quæ habeat factorem ſimplicem $x - a$, aut factorem compoſitum $(x - a)^m$, in quo *m* eſt numerus poſitivus: functio *P* evaneſcit facto $x = a$.

Viciſ-

Viciſſim ſi functio P evaneſcit, facto $x=a$; functio hæc admittit diviſores formarum præcedentium, in quibus m eſt numerus poſitivus. Æquatione $P=0$ liberata a functionibus ſurdis & transcendentibus, quas poteſt continere: inverſa hæc unum eſt ex præcipuis fundamentis theoriæ æquationum, & a mathematicis admittitur, caſu ſaltem quo m eſt numerus integer poſitivus. Plerumque vero eadem per ſe evidens fit, functiones ſurdas aut transcendentes functionis P in ſeries convertendo, aut alias adhibendo transformationes. Quod in gratiam tironum pluribus exemplis illuſtrare e re eſſe cenſeo.

Exemplum 1. Sit $P=a-\sqrt{(aa-xx)}$, quæ evaneſcit facto $x=0$; dico, functionem hanc admittere diviſorem $x-0=x$.

Etenim $P=(a-\sqrt{(aa-xx)})=\frac{(a-\sqrt{(aa-xx)})(a+\sqrt{(aa-xx)})}{a+\sqrt{(aa-xx)}}=\frac{aa-(aa-xx)}{a+\sqrt{(aa-xx)}}=\frac{xx}{a+\sqrt{(aa-xx)}}$.

Vel $\sqrt{(aa-xx)}=a-\frac{\frac{1}{2}}{1}\cdot\frac{xx}{a}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{x^4}{a^3}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^6}{a^5}-\ldots$

unde $P=\frac{\frac{1}{2}}{1}\cdot\frac{xx}{a}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{x^4}{a^3}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^6}{a^5}+\ldots$

Exemplum 2. Sit $P=\sqrt{(a+x)}-\sqrt{(a-x)}$, quæ evaneſcit facto $x=0$.

$P=\frac{(\sqrt{(a+x)}-\sqrt{(a-x)})(\sqrt{(a+x)}+\sqrt{(a-x)})}{\sqrt{(a+x)}+\sqrt{(a-x)}}=\frac{a+x-(a-x)}{\sqrt{(a+x)}+\sqrt{(a-x)}}=\frac{2x}{\sqrt{(a+x)}+\sqrt{(a-x)}}$.

Vel $\sqrt{(a+x)}=\sqrt{a}+\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{xx}{a\sqrt{a}}+\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots$

$\sqrt{(a-x)}=\sqrt{a}-\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{xx}{a\sqrt{a}}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots$

unde $P=2\cdot\frac{\frac{1}{2}}{1}\cdot\frac{x}{\sqrt{a}}+2\cdot\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\cdot\frac{x^3}{aa\sqrt{a}}-\ldots$

Exemplum 3. Sit $P=a^m-x^m$, quæ evaneſcit facto $x=a$.

Per theorema Coteſianum functio P admittit factorem $x-a$, ſi m eſt numerus integer poſitivus.

Sit autem m quivis alius numerus.

Eſt $x^m=(x-(a-x))^m=a^m-\frac{m}{1}a^{m-1}(a-x)+\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}(a-x)^2-\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}(a-x)^3+\ldots$

unde $P(=a^m-x^m)=\frac{m}{1}a^{m-1}(a-x)-\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}(a-x)^2+\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}(a-x)^3-\ldots$

Exem-

Exemplum 4. Sit $P=(a+x)^m-(a-x)^m=0$ facto $x=0$.

$$(a+x)^m=a^m+\frac{m}{1}a^{m-1}x+\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}x^2+\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3+\ldots.$$

$$(a-x)^m=a^m-\frac{m}{1}a^{m-1}x+\frac{m}{1}\cdot\frac{m-1}{2}a^{m-2}x^2-\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3+\ldots.$$

$$P\;=\;2.\frac{m}{1}a^{m-1}x\qquad\qquad+2.\frac{m}{1}\ldots\frac{m-2}{3}a^{m-3}x^3+\ldots.$$

Exemplum 5. Sit $P=\sqrt{(aa+ax+xx)}-\sqrt{(aa-ax+xx)}$, evaneſcens facto $x=0$.

$P=\dfrac{(aa+ax+xx)-(aa-ax+xx)}{\sqrt{(aa+ax+xx)}+\sqrt{(aa-ax+xx)}}=\dfrac{2ax}{\sqrt{(aa+ax+xx)}+\sqrt{(aa-ax+xx)}}$, quæ admittit factorem x.

Idem patet convertendo in ſeries quantitates $\sqrt{((a+x)^2-ax)}$, $\sqrt{((a-x)^2+ax)}$.

Exemplum 6. Sit $P=a-\sqrt[4]{a^3x}$, quæ evaneſcit facto $x=a$.

$\sqrt[4]{a^3x}=\sqrt[4]{(a^4-a^3(a-x))}$, quæ in ſeriem converſa, & ab a ſubtracta, reſiduum relinquit, cujus factor eſt $a-x$.

Pariter $P=a-\sqrt[4]{aa-xx}=a-\sqrt[4]{(a^4-aa(aa-xx))}$, in ſeriem converſa, offert factorem $aa-xx=(a-x)(a+x)$.

$P=a-\sqrt[4]{ax^3}=a-\sqrt[4]{(a^4-a(a^3-x^3))}$, in ſeriem converſa, offert factorem $a^3-x^3=(a-x)(aa+ax+xx)$.

Generatim $P=a-\sqrt[m]{a^{m-n}x^n}=a-\sqrt[m]{(a^m-a^{m-n}(a^n-x^n))}$, in ſeriem converſa, offert factorem a^n-x^n, qui admittit factorem $a-x$.

Exemplum 7. Sit $P=\sqrt{(2aaxx-x^4)}-a\sqrt[3]{aax}$, quæ evaneſcit poſito $x=a$.

$$\sqrt{(2aaxx-x^4))}=\sqrt{(a^4-(aa-xx)^2)}=aa-\frac{\frac{1}{2}}{1}\frac{(aa-xx)^2}{aa}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{(aa-xx)^4}{a^6}-\frac{\frac{1}{2}}{1}\cdot\frac{\frac{1}{2}}{2}\cdot\frac{\frac{3}{2}}{3}\frac{(aa-xx)^6}{a^{10}}-\ldots$$

$$a\sqrt[3]{aax}\;=a\sqrt[3]{a^3-aa(a-x))}=aa-\frac{\frac{1}{3}}{1}\cdot a(a-x)-\frac{\frac{1}{3}}{1}\cdot\frac{\frac{2}{3}}{2}\cdot(a-x)^2-\frac{\frac{1}{3}}{1}\cdot\frac{\frac{2}{3}}{2}\cdot\frac{\frac{5}{3}}{3}\cdot\frac{(a-x)^3}{a}-\ldots$$

Subtractione facta remanet factor $a-x$.

Exemplum 8. Sit $P=x\pm\text{ſin.}\,x=0$, quando $x=0$.

$$\text{Quoniam ſin.}\,x=\;x-\frac{1}{1.2.3}x^3+\frac{1}{1\ldots5}x^5-\frac{1}{1\ldots7}x^7+\ldots.$$

$$x+\text{ſin.}\,x=2x-\frac{1}{1.2.3}x^3+\frac{1}{1\ldots5}x^5-\frac{1}{1\ldots7}x^7-\ldots.$$

$$x-\text{ſin.}\,x=\qquad\frac{1}{1.2.3}x^3-\frac{1}{1\ldots5}x^5+\frac{1}{1\ldots7}x^7-\ldots.$$

Exem-

Exemplum 9. Sit $P = e^x - e^{-x} = 0$, quando $x = 0$.

$$P = 2x\left(\frac{1}{1.2} + \frac{1}{1.2..4}xx + \frac{1}{1...6}x^4 + \ldots\right)$$

§. 148. 1°. Sit P functio variabilis x evaneſcens caſu $x = a$, quæ igitur reduci poteſt ad formam $(x-a)P'$; non autem evaneſcat P' caſu $x = a$, nec proinde P' reduci poſſit ad formam $(x-a)P''$; ſed caſu $x = a$ determinatum obtineat valorem A. Quæritur hic valor.

Ob $(x-a)P' = P$; fit differentiando $P' + (x-a)\frac{dP'}{dx} = \frac{dP}{dx}$:

unde $A = \frac{dP}{dx}$, quando in exponente differentiali $\frac{dP}{dx}$ quantitas conſtans a loco mutabilis x ſubſtituitur. Brevitatis cauſa deſignet ${}^a\left(\frac{dP}{dx}\right)$ valorem exponentis differentialis $\frac{dP}{dx}$, quando in eo loco x quantitas conſtans a ſubſtituitur. Fit ideo $A = {}^a\left(\frac{dP}{dx}\right)$.

2°. Pariter ſit $Q = (x-a)Q'$, nec Q' evaneſcat caſu $x = a$; valor determinatus B, quem Q' recipit caſu $x = a$, eſt $B = {}^a\left(\frac{dQ}{dx}\right)$.

3°. Tum vero $\frac{P}{Q}\left(= \frac{(x-a)P'}{(x-a)Q'}\right) = \frac{P'}{Q'}$; caſu $x = a$ fit $\frac{A}{B} = \frac{{}^a\left(\frac{dP}{dx}\right)}{{}^a\left(\frac{dQ}{dx}\right)} = {}^a\left(\frac{dP}{dQ}\right)$.

Exemplum. Sit $P = x^m - a^m$, quæ evaneſcit facto $x = a$: $\frac{dP}{dx} = mx^{m-1}$, & ${}^a\left(\frac{dP}{dx}\right) = ma^{m-1} = A$. Pariter ſit $Q = x^n - a^n$, quæ evaneſcit facto $x = a$: $\frac{dQ}{dx} = nx^{n-1}$, & ${}^a\left(\frac{dQ}{dx}\right) = na^{n-1} = B$. Hinc $\frac{A}{B} = \frac{m}{n}a^{m-n}$.

§. 149. 1°. Quodſi (§. 148. 1°.) etiam P' evaneſcit caſu $x = a$; proindeque eſt $P' = (x-a)P''$; ſed P'' non evaneſcit caſu $x = a$: eodem modo conſequitur, valorem determinatum A, quem P'' caſu $x = a$ obtinet, eſſe $A = {}^a\left(\frac{dP'}{dx}\right)$.

Atqui

Atqui ob $P' + (x-a)\frac{dP'}{dx} = \frac{dP}{dx}$

eſt $\frac{2dP'}{dx} + (x-a)\frac{ddP'}{dx^2} = \frac{ddP}{dx^2}$: proinde caſu $x=a$, $2A = {}^a\left(\frac{ddP}{dx^2}\right)$, & $A = \frac{1}{1.2}\,{}^a\left(\frac{ddP}{dx^2}\right)$.

2°. Pariter ſi (§. 148. 2°.) etiam $Q' = (x-a)Q''$, nec Q'' evaneſcit facto $x=a$; valor determinatus B, quem Q'' caſu $x=a$ obtinet, eſt $B = \frac{1}{1.2}\,{}^a\left(\frac{ddQ}{dx^2}\right)$.

3°. Tum vero $\frac{P}{Q}\left(=\frac{(x-a)^2 . P''}{(x-a)^2 . Q''}\right) = \frac{P''}{Q''}$, fit $\frac{B}{A} = \frac{{}^a\left(\frac{ddP}{dx^2}\right)}{{}^a\left(\frac{ddQ}{dx^2}\right)}$ caſu $x=a$.

Exemplum. Sit $P = a^{n+1} - (n+1)ax^n + nx^{n+1}$.

$$\frac{dP}{dx} = -n.\overline{n+1}\,ax^{n-1} + n.\overline{n+1}\,x^n = 0, \text{ facto } x=a,$$

$$\frac{ddP}{dx^2} = -n.\overline{n-1}.\overline{n+1}\,ax^{n-2} + n.n.\overline{n+1}\,x^{n-1}.$$

$${}^a\left(\frac{ddP}{dx^2}\right) = n.\overline{n+1}\,a^{n-1} = 2.A.$$

Pariter ſit $Q = a^{m+1} - (m+1)ax^m + mx^{m+1}$

$${}^a\left(\frac{ddQ}{dx^2}\right) = m.\overline{m+1}\,a^{m-1} = 2B.$$

unde $\frac{A}{B} = \frac{{}^a\left(\frac{ddP}{dx^2}\right)}{{}^a\left(\frac{ddQ}{dx^2}\right)} = \frac{n.\overline{n+1}}{m.\overline{m+1}}a^{n-m}$.

§. 150. 1°. Quodſi (§. 149. 1°.) & P'' evaneſcit caſu $x=a$, & proinde eſt $P'' = (x-a)P'''$; ſed P''' non evaneſcit caſu $x=a$: eodem modo conſequitur, valorem determinatum A, quem P'' caſu $x=a$ obtinet, eſſe $A = {}^a\left(\frac{dP''}{dx}\right)$.

Sed propter $P'' + (x-a)\frac{dP''}{dx} = \frac{dP'}{dx}$

eſt $2\frac{dP''}{dx^2} + (x-a)\frac{ddP''}{dx^2} = \frac{ddP'}{dx^2}$, quare facto $x=a$, eſt $A = \frac{1}{1.2}\,{}^a\left(\frac{ddP'}{dx^2}\right)$.

et

et propter $2\frac{dP'}{dx}+(x-a)\frac{ddP'}{dx^2}=\frac{ddP}{dx^2}$

eſt $3\frac{ddP'}{dx^2}+(x-a)\frac{d^3P'}{dx^3}=\frac{d^3P}{dx^3}$, & $3\,{}^a\!\left(\frac{ddP'}{dx^2}\right)={}^a\!\left(\frac{d^3P}{dx^3}\right)$, quare $A=\frac{1}{1.2.3}\,{}^a\!\left(\frac{d^3P'}{dx^3}\right)$.

2°. Pariter ſi (§. 149. 2°.) & Q'' evaneſcit caſu $x=a$; ac proinde eſt $Q''=(x-a)Q'''$; ſed Q''' non evaneſcit caſu $x=a$: valor determinatus B, quem Q''' caſu $x=a$ obtinet, erit $B=\frac{1}{1.2.3}\,{}^a\!\left(\frac{d^3Q}{dx^3}\right)$.

3°. Tum vero $\frac{P}{Q}\left(=\frac{(x-a)^3P'''}{(x-a)^3Q'''}\right)=\frac{P'''}{Q'''}$; fit $\frac{A}{B}=\frac{{}^a\!\left(\frac{d^3P}{dx^3}\right)}{{}^a\!\left(\frac{d^3Q}{dx^3}\right)}$.

Exemplum. Sit $P=a(x-a)^3(xx+aa)$
$Q=(x-a)^3(x^3+a^3)$

$${}^a\!\left(\frac{d^3P}{dx^3}\right)=1.2.3.2a^3 \quad A=2a^3$$
$${}^a\!\left(\frac{d^3Q}{dx^3}\right)=1.2.3.2a^3 \quad B=2a^3$$
$$\frac{A}{B}=\frac{{}^a\!\left(\frac{d^3P}{dx^3}\right)}{{}^a\!\left(\frac{d^3Q}{dx^3}\right)}=\frac{2a^3}{2a^3}=1.$$

§. 151. Si rurſus (§. 150.) P''' & Q''' evaneſcunt, poſito $x=a$; & proinde ſunt $P'''=(x-a)P^{\text{IV}}$, $Q'''=(x-a)Q^{\text{IV}}$; ita tamen ut P^{IV} & Q^{IV} non evaneſcant caſu $x=a$; ſed P^{IV} & Q^{IV} hoc caſu valores habent determinatos A, B: erit

$$A={}^a\!\left(\frac{dP'''}{dx}\right),\quad B={}^a\!\left(\frac{dQ'''}{dx}\right),\ \&$$

$$A=\frac{1}{1\ldots4}\,{}^a\!\left(\frac{d^4P}{dx^4}\right),\quad B=\frac{1}{1\ldots4}\,{}^a\!\left(\frac{d^4Q}{dx^4}\right):\ \text{unde}\ \frac{A}{B}=\frac{{}^a\!\left(\frac{d^4P}{dx^4}\right)}{{}^a\!\left(\frac{d^4Q}{dx^4}\right)}.$$

§. 152. *Generatim.* Sit $P=(x-a)^mP'$, exponente m denotante numerum integrum poſitivum; & P' non ſit diviſibilis per $x-a$.

Erit $\frac{dP}{dx}=m(x-a)^{m-1}.P'$
$+\ (x-a)^m.\frac{dP'}{dx}$.

$$\frac{ddP}{dx^2} = m \cdot m-1(x-a)^{m-2}.\, P' + 2m\, (x-a)^{m-1}.\, \frac{dP'}{dx} + (x-a)^{m}.\, \frac{ddP'}{dx^2}$$

$$\frac{d^3P}{dx^3} = m \ldots m-2(x-a)^{m-3}.\, P' + 3m \cdot m-1\, (x-a)^{m-2}.\, \frac{dP'}{dx} + 3m\, (x-a)^{m-1}.\, \frac{ddP'}{dx^2} + (x-a)^{m}.\, \frac{d^3P'}{dx^3}$$

$$\frac{d^4P}{dx^4} = m \ldots m-3\,(x-a)^{m-4}.\, P' + 4m \ldots m-2\,(x-a)^{m-3}.\, \frac{dP'}{dx} + 6m \ldots m-2\,(x-a)^{m-2}.\, \frac{ddP'}{dx^2} + 4m\, (x-a)^{m-1}.\, \frac{d^3P'}{dx^3} + (x-a)^{m}.\, \frac{d^4P'}{dx^4}$$

$$\vdots$$

$$\frac{d^nP}{dx^n} = m \ldots m-(n-1).(x-a)^{m-n}.\, P' + \frac{n}{1} m \ldots m-(n-2).(x-a)^{m-(n-1)}.\, \frac{dP'}{dx} + \frac{n}{1} \cdot \frac{n-1}{2} m \ldots m-(n-3).(x-a)^{m-(n-2)}.\, \frac{ddP'}{dx^2} + \frac{n}{1} \ldots \frac{n-2}{3} m \ldots m-(n-4).(x-a)^{m-(n-3)}.\, \frac{d^3P'}{dx^3} \;\vdots\; + \frac{n}{1} \cdot m\, (x-a)^{m-1}.\, \frac{d^{n-1}P'}{dx^{n-1}} + (x-a)^{m}.\, \frac{d^nP}{dx^n}.$$

d^mP

$$\frac{d^m P}{dx^m} = \qquad m \ldots 1 \,.\, P'$$
$$+ \quad \frac{m}{1} \cdot m \ldots 2\,(x-a) \,.\, \frac{dP'}{dx}$$
$$+ \frac{m}{1} \cdot \frac{m-1}{2} \cdot m \ldots 3\,(x-a)^2 \,.\, \frac{ddP'}{dx^3}$$
$$+ \frac{m}{1} \cdot \frac{m-2}{3} \cdot m \ldots 4\,(x-a)^3 \,.\, \frac{d^3P'}{dx^3}$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$+ \frac{m}{1} \cdot m \qquad (x-a)^{m-1} \,.\, \frac{d^{m-1}P'}{dx^{m-1}}$$
$$+ \qquad\qquad (x-a)^m \,.\, \frac{d^m P'}{dx^m}.$$

Proinde $A = \frac{1}{1.2\ldots m}\, {}^{a}\!\left(\frac{d^m P}{dx^m}\right)$.

Pariter ſi $Q = (x-a)^m Q'$,

$$B = \frac{1}{1.2\ldots m}\, {}^{a}\!\left(\frac{d^m Q}{dx^m}\right) : \text{ unde } \frac{A}{B} = \frac{{}^{a}\!\left(\frac{d^m P}{dx^m}\right)}{{}^{a}\!\left(\frac{d^m Q}{dx^m}\right)}.$$

§. 153. Sit nunc $P = (x-a)^{\frac{1}{n}} p$, denotante n numerum integrum poſitivum. Fit ideo

$$P^n = (x-a)p^n,$$
$$\frac{dP^n}{dx} = p^n + n(x-a)p^{n-1}\frac{dp}{dx}:$$

unde $A = \sqrt[n]{\left({}^{a}\!\left(\frac{dP^n}{dx}\right)\right)}$.

Sit pariter $Q = (x-a)^{\frac{1}{n}} q$: erit $Q^n = (x-a)q^n$, & $B = \sqrt[n]{\left({}^{a}\!\left(\frac{dQ^n}{dx}\right)\right)}$:

$$\text{unde } \frac{A}{B} = \sqrt[n]{\left[\frac{{}^{a}\!\left(\frac{dP^n}{dx}\right)}{{}^{a}\!\left(\frac{dQ^n}{dx}\right)}\right]}.$$

§. 154. Sit tandem $P = (x-a)^{\frac{m}{n}} p$, ideoque $P^n = (x-a)^m p^n$:
erit ${}^{a}\!\left(\frac{d^m P^n}{dx^m}\right) = 1 \ldots m A^n$.

Sit

Sit quoque $Q = (x-a)^{\frac{m}{n}} q$; ideoque $Q^n = (x-a)^m q^n$: erit ${}^{a}\left(\frac{d^m Q^n}{dx^m}\right) = 1 \ldots m B^n$;

unde $\frac{A^n}{B^n} = \frac{{}^{a}\left(\frac{d^m P^n}{dx^m}\right)}{{}^{a}\left(\frac{d^m Q^n}{dx^m}\right)}$, & $\frac{A}{B} = \sqrt[n]{\left[\frac{{}^{a}\left(\frac{d^m P^n}{dx^m}\right)}{{}^{a}\left(\frac{d^m Q^n}{dx^n}\right)}\right]}$.

Exempla. Sit $P = \sqrt{(xx-aa)} - (x-a) = \sqrt{(x-a)}(\sqrt{(x+a)} - \sqrt{(x-a)})$; proinde $p = \sqrt{(x+a)} - \sqrt{(x-a)}$.

Erit $PP = 2(x-a)(x - \sqrt{(xx-aa)})$

$$\frac{dPP}{dx} = (x - \sqrt{xx-aa}) + 2(x-a)\left(1 - \frac{x}{\sqrt{xx-aa}}\right) = 2(x - \sqrt{(xx-aa)} + 2(x-a) - 2\sqrt{\frac{x-a}{x+a}}):$$

unde $AA = 2a$, $A = \sqrt{2a}$.

Sit $Q = \sqrt{(x^3-a^3)} - (x-a)\sqrt{x} = \sqrt{(x-a)}(\sqrt{(xx+ax+aa)} - \sqrt{x(x-a)})$

$$QQ = (x-a)(2xx+aa-2\sqrt{(x(x-a)(xx+ax+aa))}$$

$$\frac{dQQ}{dx} = (xx+aa-2\sqrt{(x(x-a)(xx+ax+aa))} + (x-a)\left(4x - \frac{(x-a)(xx+ax+aa)+x(xx+ax+aa)+x(x-a)(2x+a)}{\sqrt{x(x-a)(xx+ax+aa))}}\right)$$

$$= (xx+aa-2\sqrt{(x(x-a)(xx+ax+aa))} + (x-a)4x - \sqrt{(x-a)}\frac{(x-a)(xx+ax+aa)+x(xx+ax+aa)+x(x-a)(2x+a)}{\sqrt{(x(xx+ax+aa))}}$$

proinde $BB = 2aa$, $B = a\sqrt{2}$.

hinc $\frac{A}{B} = \frac{\sqrt{2a}}{a\sqrt{2}} = \frac{1}{\sqrt{a}}$.

§. 155. *Obſervatio.* Ex formula (§. 152.)

$$\begin{aligned}\frac{d^m P}{dx^m} = {} & \quad m \ldots 1 P' \\ & + \frac{m}{1} \cdot m \ldots 2 (x-a) \frac{dP'}{dx} \\ & + \frac{m}{1} \cdot \frac{m-1}{2} \cdot m \ldots 3 (x-a)^2 \frac{ddP'}{dx^2} \\ & + \frac{m}{1} \cdots \frac{m-2}{3} \cdot m \ldots 4 (x-a)^3 \frac{d^3 P'}{dx^3} \\ & \vdots \\ & + \frac{m}{1} \cdot m \quad (x-a)^{m-1} \frac{d^{m-1} P'}{dx^{m-1}} \\ & + \quad (x-a)^m \frac{d^m P'}{dx^m}\end{aligned}$$

ſequitur, quantitatem $m \ldots 1 A$,

quæ

quæ eſt limes poſterioris membri hujus æquationis, limitem eſſe etiam exponentis differentialis $\frac{d^mP}{dx^m}$, ſeu $A = \frac{1}{1.2...m}\lim.\frac{d^mP}{dx^m}$.

$$\text{Pariter } B = \frac{1}{1.2...m}\lim.\frac{d^mQ}{dx^m}; \text{ unde } \frac{A}{B} = \frac{\lim.\frac{d^mP}{dx^m}}{\lim.\frac{d^mQ}{dx^m}} = \lim.\frac{\frac{d^mP}{dx^m}}{\frac{d^mQ}{dx^m}}.$$

Hanc obſervationem paucis exemplis illuſtrare haud alienum abs re erit.

$$\text{Sit } \frac{P}{Q} = \frac{\text{ſin.}\,x}{x};\quad \begin{matrix}\frac{dP}{dx} = \text{coſ.}\,x \\ \frac{dQ}{dx} = 1.\end{matrix}\quad \lim.\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \lim.\frac{\text{coſ.}\,x}{1} = 1.$$

$$\text{Sit } \frac{P}{Q} = \frac{\text{tang.}\,x}{x},\quad \begin{matrix}\frac{dP}{dx} = \text{ſec.}^2x \\ \frac{dQ}{dx} = 1\end{matrix}\quad \lim.\frac{\frac{dP}{dx}}{\frac{dQ}{dx}} = \lim.\text{ſec.}^2x = 1.$$

Sit S ſumma progreſſionis geometricæ $S = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$: eſt $S = \frac{1-x^n}{1-x} = \frac{x^n-1}{x-1}$, prouti progreſſio decreſcit, aut creſcit inde ab unitate. Quodſi autem ſeries propoſita neque creſcit neque decreſcit; ſeu ſi ſeries propoſita terminis conſtet inter ſe æqualibus: fit $S = \frac{1-1^n}{1-1} = \frac{1-1}{1-1}$.

Dico autem: methodos, quibus progreſſionum geometricarum ſummæ inveſtigantur, ad hunc caſum non poſſe applicari.

Prima methodus eo redit, ut inferatur (ex *Prop.* 12. *Lib. V. Elem.*): ſumma omnium terminorum excepto ultimo eſt ad ſummam omnium terminorum excepto primo, uti primus terminus ad ſecundum. Unde, caſu æqualitatis omnium terminorum, fit $S-1 : S-1 = 1 : 1$, $S-1 = S-1$; quæ eſt æquatio identica, ex qua nihil concludi poteſt: & ſi computus ulterius continuetur, fit $S(1-1) = 1-1$, $S = \frac{1-1}{1-1}$, quæ eſt expreſſio indeterminata.

Transeo ad alteram methodum.

$$\begin{array}{lll}
\text{Sit} & S = 1 + x + x^2 + x^3 + x^4 + \ldots x^{n-1} & \\
\text{erit} & Sx = \quad x + x^2 + x^4 + x^4 + \ldots x^{n-1} + x^n & \\
\text{unde} & S(x-1) = -1 & +x^n \text{ (ſi progreſſio creſcit)} \\
 & S(1-x) = 1 & -x^n \text{ (ſi progreſſio decreſcit)}
\end{array}$$

 Sit

Sit autem ſeries neque creſcens neque decreſcens: erit

$$\begin{aligned} S &= 1 + 1 + 1 + 1 + \ldots + 1 \\ S.1 &= \quad\;\; 1 + 1 + 1 + \ldots + 1 + 1 \\ S(1-1) &= 1 \qquad\qquad\qquad\qquad\quad - 1 \end{aligned} \qquad S = \frac{1-1}{1-1},$$

quæ iterum eſt expreſſio indeterminata.

Cum autem ratio æqualitatis limes ſit tam rationis decreſcentis $x : 1$ (poſito $x > 1$), quam rationis creſcentis $x : 1$, poſito $x < 1$; pariter ſumma terminorum ſibi invicem æqualium tam limes eſt parvitatis ſummæ decreſcentis totidem terminorum in progreſſione geometrica a primo inde prioribus æquali creſcentium, quam limes magnitudinis ſummæ creſcentis totidem terminorum in progreſſione geometrica a primo inde prioribus æquali decreſcentium. Unde ſumma terminorum ſibi invicem æqualium (quamvis ſeriem geometricam non conſtituant) elici poteſt ex ſumma totidem terminorum ab eodem primo termino in ſerie geometrica progredientium; quærendo ſcilicet limitem ſummæ hujus progreſſionis, quatenus exponens progreſſionis ad unitatem accedit.

Eſt nempe

$$S = \frac{x^n - 1}{x - 1}. \quad \text{Sit } x = 1 + z; \; \frac{x^n - 1}{x - 1} = \frac{n}{1} + \frac{n}{1} \cdot \frac{n-1}{2} z + \frac{n}{1} \ldots \frac{n-2}{3} z^2 + \frac{n}{1} \ldots \frac{n-3}{4} z^3 + \ldots .$$

cujus ſeriei limes parvitatis eſt n. Sed hic limes eſt ſeries totidem terminorum priori æqualium: ergo ſumma n terminorum unitati æqualium eſt n.

Pariter ſit

$$\begin{aligned} S &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots + n \quad x^{n-1} \\ Sx &= \quad\;\; x + 2x^2 + 3x^3 + 4x^4 + \ldots + n-1 . x^{n-1} + n x^n \\ S - Sx &= 1 + x + x^2 + x^3 + x^4 + \ldots + \quad x^{n-1} - n x^n \\ Sx - Sx^2 &= \quad\;\; x + x^2 + x^3 + x^4 + \ldots + \quad x^{n-1} + \; x^n - n x^{n+1} \\ S - 2Sx + Sx^2 &= 1 \qquad\qquad - (n+1) x^n + n x^{n+1} = n(x^{n+1} - x^n) - (x^n - 1). \end{aligned}$$

Facto $x = 1$, erit $S((1-1)-(1-1) = (n-n)-(1-1))$; ex qua expreſſione nihil deducere licet.

Fiat autem $x = 1 + z$, & quæratur limes expreſſionis $\frac{n(x^{n+1} - x^n) - (x^n - 1)}{(x-1)^2}$: erit

$S =$

$$S = {}^{n}\frac{\left\{\begin{array}{l} 1+\frac{n+1}{1}z+\frac{n+1}{1}\cdot\frac{n}{2}z^2+\frac{n+1}{1}\ldots\frac{n-1}{3}z^3+\ldots \\ -(1+\frac{n}{1}-z+\frac{n}{1}\cdot\frac{n-1}{2}z^2+\frac{n}{1}\ldots\frac{n-2}{3}z^3+\ldots \end{array}\right. -\left(\frac{n}{1}z+\frac{n}{1}\cdot\frac{n-1}{2}z^2+\frac{n}{1}\ldots\frac{n-2}{3}z^3+\ldots\right)}{zz}$$

$$=\frac{n\cdot n+1}{1.2}+2.\frac{n-1\cdot n\cdot n+1}{1.2.3}z+3.\frac{n-2\cdot n-1\ldots n+1}{1\ldots 4}z^2+\ldots$$ cujus limes eſt $\frac{n\cdot n+1}{1.2}$.

Eademque computandi methodus ad plurimas ſeries applicatur, numerorum nempe figuratorum, & generatim terminorum ad differentias conſtantes perducentium, eorundemque per terminos ſeriei alicujus geometricæ multiplicatorum.

§. 156. Hinc intelligitur, quomodo theorema Taylorianum poſſit ad hanc inveſtigationem adhiberi. Scilicet ſit $P' = \frac{P}{x-a}$; ſitque P' functio integra ipſius x. Sit Δx valor ipſius $x-a$, præcedens valorem $x-a=0$: valor ipſius P huic valori Δx reſpondens erit $\frac{\Delta x}{1}\frac{dP}{dx}+\frac{\Delta x^2}{1.2}\frac{ddP}{dx^2}+\frac{\Delta x^3}{1.2.3}\frac{d^3P}{dx^3}+\ldots$;

proinde $$P' = \frac{dP}{dx}+\frac{\Delta x}{1.2}\frac{ddP}{dx^2}+\frac{\Delta x^2}{1.2.3}\frac{d^3P}{dx^3}+\ldots,$$ cujus limes eſt ${}^{a}\left(\frac{dP}{dx}\right)$.

Eodemque modo $$Q' = \frac{dQ}{dx}+\frac{\Delta x}{1.2}\frac{ddQ}{dx^2}+\frac{\Delta x^2}{1.2.3}\frac{d^3Q}{dx^3}+\ldots,$$ cujus limes eſt ${}^{a}\left(\frac{dQ}{dx}\right)$:

unde $\frac{A}{B} = {}^{a}\left(\frac{dP}{dQ}\right)$.

Eodemque modo ſi ${}^{a}\left(\frac{dP}{dx}\right)=0$, & ${}^{a}\left(\frac{dQ}{dx}\right)=0$: erit $\frac{A}{B}=\frac{{}^{a}\left(\frac{ddP}{dx^2}\right)}{{}^{a}\left(\frac{ddQ}{dx^2}\right)}$, & ſic deinceps; quousque ad exponentes differentiales ſimul non evaneſcentes perveniatur.

§. 157. Huc pertinet determinatio (ſi poſſibilis ſit) differentiæ expreſſionum infiniti ſeu impoſſibilis, quæ prodeunt, dum operationes quædam ultra caſus, ad quos quadrant, extenduntur; ſeu expreſſiones $\frac{1}{0}-\frac{1}{0}$, $\infty-\infty$, $(1-1)\infty$, $0\times\infty$, $0\times\frac{1}{0}$; quippe quæ ita redeunt ad $\frac{0}{0}$.

Exempli loco ſit applicatio reſolutionis fractionum, quarum denominatores

 facto-

factores habent a ſe invicem diverſos, ad caſum, quo factores hi fiunt inter ſe æquales.

Nempe $\frac{1}{x-a}\cdot\frac{1}{x-a'} \begin{matrix} =\frac{1}{a-a'}\cdot\frac{1}{x-a} \\ +\frac{1}{a'-a}\cdot\frac{1}{x-a'} \end{matrix} = \frac{1}{a-a'}\left(\frac{1}{x-a}-\frac{1}{x-a'}\right)$; facto autem $a=a'$,

ſignum impoſſibilis $\frac{1}{a-a'}$ nos monet, reſolutionem propoſitam eſſe impoſſibilem; & quærenda eſt differentia duarum expreſſionum impoſſibilium $\frac{1}{a-a'}\cdot\frac{1}{x-a}$, $\frac{1}{a-a'}\cdot\frac{1}{x-a'}$.

Eſt autem $\frac{1}{x-a'} = \frac{1}{x-a+(a-a')} = \frac{1}{x-a} - \frac{a-a'}{(x-a)^2} + \frac{(a-a')^2}{(x-a)^3} - \frac{(a-a')^3}{(x-a)^4} + \ldots\ldots$

$$\frac{1}{a-a'}\left(\frac{1}{x-a}-\frac{1}{x-a'}\right) = \frac{1}{a-a'}\left(\frac{a-a'}{(x-a)^2} - \frac{(a-a')^2}{(x-a)^3} + \frac{(a-a')^3}{(x-a)^4} - \ldots\ldots\right)$$

$$= \frac{1}{(x-a)^2} - \frac{a-a'}{(x-a)^3} + \frac{(a-a')^2}{(x-a)^4} - \ldots = \frac{1}{(x-a)^2}$$ caſu, quo $a=a'$.

Pariter $\frac{1}{\overline{x-a}\,.\,\overline{x-a'}\,.\,\overline{x-a''}} = \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\cdot\frac{1}{x-a}$

$$+\frac{1}{\overline{a'-a}\,.\,\overline{a'-a''}}\cdot\frac{1}{x-a'}$$

$$+\frac{1}{\overline{a''-a}\,.\,\overline{a''-a'}}\cdot\frac{1}{x-a''} = \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{x-a} - \frac{a-a''}{x-a'} + \frac{a-a'}{x-a''}\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{x-a} - \frac{a'-a''}{x-a'} - \left(\frac{a-a'}{x-a'} - \frac{a-a'}{x-a''}\right)\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{1}{x-a} - \frac{1}{x-a'}\right) - \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{1}{x-a'} - \frac{1}{x-a''}\right) = \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{1}{(x-a')-(a-a')} - \frac{1}{x-a'}\right) - \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{1}{x-a'} - \frac{1}{(x-a')+(a'-a'')}\right)$$

$$= \frac{1}{\overline{a-a'}\,.\,\overline{a-a''}}\left(\frac{a-a'}{(x-a')^2} + \frac{(a-a')^2}{(x-a')^3} + \frac{(a-a')^3}{(x-a')^4} + \frac{(a-a')^4}{(x-a')^5} + \ldots\ldots\right)$$

$$- \frac{1}{\overline{a-a''}\,.\,\overline{a'-a''}}\left(\frac{a'-a''}{(x-a')^2} - \frac{(a'-a'')^2}{(x-a')^3} + \frac{(a'-a'')^3}{(x-a')^4} - \frac{(a'-a'')^4}{(x-a')^5} + \ldots\ldots\right)$$

$=$

$$= \frac{1}{a-a''}\left(\frac{1}{(x-a')^2} + \frac{a-a'}{(x-a')^3} + \frac{(a-a')^2}{(x-a')^4} + \frac{(a-a')^3}{(x-a')^5} + \ldots\right)$$

$$- \frac{1}{a-a''}\left(\frac{1}{(x-a')^2} - \frac{a'-a''}{(x-a')^3} + \frac{(a'-a'')^2}{(x-a')^4} - \frac{(a'-a'')^3}{(x-a')^5} + \ldots\right)$$

$$= \frac{1}{a-a''}\left(\qquad \frac{a-a''}{(x-a')^3} + \frac{(a-a'')(a-2a'+a'')}{(x-a')^4} + \frac{(a-a'')((a-a')^2-(a-a')(a'-a'')+(a'-a'')^2)}{(x-a')^5} + \ldots\right)$$

$$= \frac{1}{(x-a')^3} + \frac{a-2a'+a''}{(x-a')^4} + \ldots$$

Unde poſito $a = a' = a''$, fit

$$\left.\begin{array}{l} \frac{1}{a-a'.a-a''}\cdot\frac{1}{x-a} \\ + \frac{1}{a'-a.a'-a''}\cdot\frac{1}{x-a'} \\ + \frac{1}{a''-a.a''-a'}\cdot\frac{1}{x-a''} \end{array}\right\} = \frac{1}{(x-a')^3} = \frac{1}{(x-a)^3}.$$

Expulſis igitur impoſſibilitatis ſignis, coacta calculorum ad caſus, ad quos non quadrant, applicatione introductis, reſtituitur expreſſio realis $\frac{1}{(x-a)^3}$, quæ in expreſſiones impoſſibiles fuerat reſoluta.

Idem dicatur de reſolutione impoſſibili fractionum $\frac{1}{(x-a)^n}$, $\frac{1}{(xx-2ax\,\mathrm{coſ}.\alpha+aa)^n}$. (a)

§. 158. Progredior ad nonnulla exempla, quibus differentiæ propoſitæ quantitates transcendentes comprehendunt; eaque deſumta ex Euleri *Inſtitutionum calculi differentialis* parte poſteriori Cap. XV.

Exemplum primum. Sit functio $\frac{x}{x-1} - \frac{1}{\log.x}$, cujus termini $\frac{x}{x-1}$ & $\frac{1}{\log.x}$ fiunt $\frac{1}{0}$ ſeu ∞ facto $x = 1$.

Sit $\frac{x}{x-1} - \frac{1}{\log.x} = \frac{x\log.x-(x-1)}{x-1.\log.x} = \frac{P}{Q}$.

 Erit

(a) Applicationes inter huc usque traditorum notari inprimis meretur fractionum rationalium, quarum denominator factores habet tam ſimplices quam compoſitos, tam reales quam imaginarios, in alias reſolutio a celeb. Eulero in *Inſtitutionibus calculi differentialis* Cap. XVIII. expoſita. Quamvis enim reſolutio hæc methodis mere elementaribus (etiam abſque methodo indeterminatarum) generaliter inſtitui poſſit; propoſitiones capite hoc ſtabilitas mire eam juvare ac concinniorem reddere non eſt diffitendum.

Erit $\frac{dP}{dx} = \log.x + 1 - 1 = \log.x = 0$, facto $x = 1$

$$\frac{dQ}{dx} = \log.x + \frac{x-1}{x} \qquad = 0, \text{ facto } x = 1$$

$$\frac{ddP}{dx^2} = \frac{1}{x} \qquad = 1, \text{ facto } x = 1$$

$$\frac{ddQ}{dx^2} = \frac{1}{x} + \frac{1}{x} - \frac{x-1}{xx} \qquad = 2, \text{ facto } x = 1.$$

Hinc $\frac{1\left(\frac{ddP}{dx^2}\right)}{1\left(\frac{ddQ}{dx^2}\right)} = \frac{A}{B} = \frac{1}{2}$.

Exemplum ſecundum. Sit functio $\frac{1}{x(e^{2x}-1)} - \frac{1-x}{2xx}$, cujus quæritur valor caſu, quo $x = 0$; & proinde uterque terminus impoſſibilis.

$$2\frac{P}{Q} = \frac{(x+1)-(1-x)(e^{2x})}{xx(e^{2x}-1)}.$$

$$\frac{dP}{dx} = 1 + (2x-1)e^{2x} \qquad = 0, \text{ facto } x = 0$$

$$\frac{dQ}{dx} = 2x(e^{2x}-1) + 2xxe^{2x} \qquad = 0, \text{ facto } x = 0$$

$$\frac{ddP}{dx^2} = 2xe^{2x} \qquad = 0, \text{ facto } x = 0$$

$$\frac{ddQ}{dx^2} = 2(e^{2x}-1) + 8xe^{2x} + 4xxe^{2x} = 0, \text{ facto } x = 0$$

$$\frac{d^3P}{dx^3} = 4e^{2x} + 4xe^{2x} \qquad = 4, \text{ facto } x = 0$$

$$\frac{d^3Q}{dx^3} = 12e^{2x} + 24xe^{2x} + 8xxe^{2x} = 12, \text{ facto } x = 0.$$

Hinc $\frac{0\left(\frac{d^3P}{dx^3}\right)}{0\left(\frac{d^3Q}{dx^3}\right)} = \frac{A}{B} = \frac{4}{12} = \frac{1}{3}$.

Aliter $e^{2x}-1 = 2x+\frac{2^2}{1.2}x^2+\frac{2^3}{1.2.3}x^3+\frac{2^4}{1\ldots4}x^4+\ldots.$

hinc $\frac{1}{x(e^{2x}-1)}-\frac{1-x}{2xx}=\frac{1}{2xx}\left\{\frac{1}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}xx+\frac{2^3}{1\ldots4}x^3+\ldots}-(1-x)\right.$

$$=\frac{1}{2xx}\left\{\frac{x(1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1\ldots4}x^3+\ldots) -2x(\frac{1}{1.2}+\frac{2}{1.2.3}x+\frac{2^2}{1\ldots4}x^2+\frac{2^3}{1\ldots5}x^3+\ldots)}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1\ldots4}x^3+\ldots)}\right.$$

$$=\frac{1}{2xx}\left\{\frac{xx(\frac{1}{3}+(\frac{2^2}{1.2.3}-\frac{2^3}{1\ldots4})x+(\frac{2^3}{1\ldots4}-\frac{2^4}{1\ldots5})x^2+\ldots}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1\ldots4}x^3+\ldots}\right.$$

$$=\frac{1}{2}\cdot\frac{\frac{1}{3}+(\frac{2^2}{1.2.3}-\frac{2^3}{1\ldots4})x+(\frac{2^3}{1\ldots4}-\frac{2^4}{1\ldots5})x^2+\ldots}{1+\frac{2}{1.2}x+\frac{2^2}{1.2.3}x^2+\frac{2^3}{1\ldots4}x^3+\ldots}=\frac{1}{6},$$

facto $x=0$.

Exemplum tertium. Sit $\frac{1}{xx}-\frac{1}{x\,\text{tang.}x}$ functio ex duobus terminis impoſſibilibus compoſita, facto $x=0$.

Ergo $\frac{P}{Q}=\frac{\text{tang.}x-x}{xx\,\text{tang.}x}$

$$\frac{dP}{dx}=\text{ſec.}^2x-1 \qquad =0,\ \text{facto } x=0$$

$$\frac{dQ}{dx}=2x\,\text{tang.}x+xx\,\text{ſec.}^2x \qquad =0,\ \text{facto } x=0$$

$$\frac{ddP}{dx^2}=2\,\text{ſec.}^2x\,\text{tang.}x \qquad =0,\ \text{facto } x=0$$

$$\frac{ddQ}{dx^2}=2\,\text{tang.}x+4x\,\text{ſec.}^2x+2xx\,\text{ſec.}^2x\,\text{tang.}x \qquad =0,\ \text{facto } x=0$$

$$\frac{d^3P}{dx^3}=4\,\text{ſec.}^2x\,\text{tang.}^2x+2\,\text{ſec.}^4x \qquad =2,\ \text{facto } x=0$$

$$\frac{d^3Q}{dx^3}=6\,\text{ſec.}^2x+12x\,\text{ſec.}^2x\,\text{tang.}x+4xx\,\text{ſec.}^2x\,\text{tang.}^2x+2xx\,\text{ſec.}^4x \qquad =6,\ \text{facto } x=0$$

Unde $\frac{A}{B}=\frac{1}{3}$, facto $x=0$.

Aliter

Aliter $\frac{1}{\text{tang.}\,x} = \frac{\text{coſ.}\,x}{\text{ſin.}\,x} = \frac{1 - \frac{1}{1.2}x^2 + \frac{1}{1\ldots4}x^4 - \frac{1}{1\ldots6}x^6 + \ldots}{x - \frac{1}{1\ldots3}x^3 + \frac{1}{1\ldots5}x^5 - \frac{1}{1\ldots7}x^7 + \ldots}$

hinc $\frac{P}{Q} = \frac{1}{xx}\left(1 - \frac{1 - \frac{1}{1.2}xx + \frac{1}{1\ldots4}x^4 - \frac{1}{1\ldots6}x^6 + \ldots}{1 - \frac{1}{1\ldots3}xx + \frac{1}{1\ldots5}x^4 - \frac{1}{1\ldots7}x^6 + \ldots}\right)$

$= \frac{\frac{1}{3} - \frac{4}{1\ldots5}xx + \frac{6}{1\ldots7}x^4 - \ldots}{1 - \frac{1}{1.2.3}xx + \frac{1}{1\ldots5}x^4 - \ldots} = \frac{1}{3}$, facto $x = 0$.

Exemplum quartum. Sit $\frac{1}{2x} - \frac{1}{x(e^x + 1)}$ functio ex duobus terminis impoſſibilibus compoſita, facto $x = 0$.

Igitur $2\frac{P}{Q} = \frac{e^x - 1}{x(e^x + 1)}$

$\frac{dP}{dx} = e^x \qquad = 1$, facto $x = 0$

$\frac{dQ}{dx} = e^x + 1 + xe^x = 2$, facto $x = 0$:

hinc $2\frac{P}{Q} = \frac{1}{2}$, facto $x = 0$, ſeu $\frac{P}{Q} = \frac{1}{4}$.

Aliter $e^x + 1 = 2 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots$

hinc $\frac{1}{2x} - \frac{1}{x(e^x + 1)} = \frac{1}{2x}\left[\frac{x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots}{2 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots}\right] = \frac{1}{2}\left[\frac{1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \ldots}{2 + x + \frac{x^2}{1.2} + \ldots}\right] = \frac{1}{4}$,

facto $x = 0$.

§. 159. Caſibus explicatis, quibus quoti $\frac{A}{B}$ fiunt determinati; paucis perſtringam caſus (non ſatis hactenus obſervatos), quibus ſignum $\frac{0}{0}$ abſolutam indicat indeterminationem. Quod ut luculentius faciam, ab exemplis quam ſimpliciſſimis ordiar.

Exem-

Exemplum primum. Sit $m : m'$ ratio data duarum quantitatum x, y; & ſit $n : n'$ ratio etiam data earundem, poſtquam quantitatibus datis a, b auctæ fuerunt.

$$\text{Erit } x = m \times \frac{n'a - nb}{nm' - n'm} \qquad x + a = n \times \frac{m'a - mb}{nm' - n'm}$$

$$y = m' \times \frac{n'a - nb}{nm' - n'm} \qquad x + b = n' \times \frac{m'a - mb}{nm' - n'm}.$$

Jam vero ſit $a : b = m : m'$.

Quoniam $x : y = m : m'$: erit $x + a : y + b = m : m'$; ideoque $n : n' = m : m'$.

$$\text{Ergo} \quad \begin{matrix} n'a = nb & x = a \times \frac{0}{0} & x + a = a \times \frac{0}{0} \\ m'a = mb & y = b \times \frac{0}{0} & y + b = b \times \frac{0}{0} \end{matrix}$$

Atqui ratione quantitatum a & b rationi quantitatum x & y æquali poſita, patet: quantitates x & y eſſe poſſe qualeſcunque, ſeu eas eſſe indeterminatas; proinde hoc caſu ſignum $\frac{0}{0}$ abſolutam indicat indeterminationem.

Caſibus, quibus quantitates x & y ſunt determinatæ, quantitas x (v. gr.) determinatur æquatione $mn'x + mn'a = nm'x + nmb$.

Atqui poſito $m : m' = n : n'$, eſt $mn' = m'n$; ideoque termini $mn'x$, $nm'x$ ſunt invicem æquales: unde debet eſſe $mn'a = nmb$, ſeu $n'a = nb$, & $a : b = n : n'$: duæ expreſſiones $mn'x + mn'a$, $nm'x + nmb$ ſunt ideo identicæ, & nihil ex illis deduci poteſt, quod ad valorem quantitatis incognitæ x, caſu, quo $m : m' = n : n' = a : b$.

Exemplum ſecundum. Proponantur duæ æquationes $\begin{matrix} ax + by = p \\ a'x + b'y = p' \end{matrix}$: fit $x = \frac{pb' - p'b}{ab' - a'b}$, $y = \frac{pa' - p'a}{a'b - ab'}$. Quamdiu non eſt $ab' = a'b$, ſeu $a : a' = b : b$; duæ quantitates incognitæ x, y æquationibus præcedentibus determinantur: caſu autem, quo $a : a' = b : b'$, ſeu $a' = na$, & ſimul $b' = nb$; fit $\begin{matrix} ax + by = p \\ nax + nby = p' \end{matrix}$: unde $p' = np$, & $p'b = nbp = pb'$; fit ideo $x = \frac{0}{0}$, ſeu quantitas indeterminata: & ſi non eſſet $p' = np$; ſigni $x = \frac{pb' - p'b}{ab \cdot 0}$ introductio quæſtionis propoſitæ impoſſibilitatem nos doceret.

Eadem applicari poſſunt ad quæſtiones, in quibus tres aut plures quantitates incognitæ occurrunt.

Exemplum tertium. Exemplo primo prorſus analogum eſt hoc alterum.

 Detur

Detur ſumma a quantitatum x, x'; it m ſumma b quantitatum y, y'; dentur etiam rationes tam quantitatum x & y, quam quantitatum x', y': & quærantur quatuor hæ quantitates. Si ratio ſummarum datarum a, b æqualis eſt rationi datæ quantitatum x, y: ratio quantitatum x', y' exinde ita determinatur, ut æqualis ſit priori rationi datæ quantitatum x, y; & hæ quantitates poſſunt eſſe quæcunque in ea ratione data, quæ indeterminatio ſymboli $\frac{0}{0}$ introductione indicatur.

Quantitates datæ numero pari a, b, c, d .... ſint ſummæ aut differentiæ quantitatum x, x', y, y', z, z', v, v' .... Dentur etiam rationes $x:y$, $y':z$, $z':v$, $v':x'$. Fieri poteſt, ut quæſtio ſit indeterminata.

Inter applicationes geometricas propoſitionis hujus ad indeterminationem ducentis tres ſequentes indicabo, quæ quaſi ſponte ſe mihi obtulerunt.

Figuræ rectilineæ poſitione ac magnitudine datæ inſcribenda ſit figura rectilinea cognominis perimetri omnium minimæ. Caſus, quo figuræ propoſitæ numerus laterum eſt par, ita eſt indeterminatus; ut, ſi problema propoſitum poſſibile fuerit, ſolutionum numerus nullum habeat limitem: & facile eſt indeterminationis hujus nexum cum caſu præcedente oſtendere. (Vid. *Relatio mutua capacitatis & terminorum figurarum.* Varſaviæ 1782.)

Pariter circulo dato inſcribenda ſit figura rectilinea, cujus latera per puncta poſitione data tranſeant. Problema hoc caſibus quibusdam etiam ſit indeterminatum; uti oſtendi in diſſertatione, quam de hoc eleganti problemate ad Academiam Berolinenſem ante aliquot menſes transmiſi: & indeterminatio hæc ad eundem omnino fontem poteſt reduci.

Idem contingit, quando figura rectilinea cognomini figuræ rectilineæ ita inſcribenda eſt, ut latera ejus per puncta poſitione data tranſeant, aut ſint rectis poſitione datis parallela.

Sed miſſis hisce quæſtionibus magis arduis transeo ad exempla geometrica ſimplicitate ſua commendanda.

Sit circulus, cui inſcribenda ſit linea recta, quæ per punctum poſitione datum tranſeat.

Sit

Sit C centrum circuli, cujus radius ſit r; ſit P punctum poſitione datum, cujus diſtantia CP a centro ſit a; ſit XY recta magnitudine data inſcribenda $= 2b$; & ſit CZ ipſi XY perpendicularis. Omiſſa analyſi & conſtructione geometrica (quibus hic immorari a ſcopo foret alienum) exponam tantum calculum algebraicum, quo fit ſin. $CPZ = \frac{CZ}{CP} = \frac{\sqrt{(rr-bb)}}{a}$. Fiat autem $b=r$, & proinde recta inſcribenda transeat per centrum; fit ſin. $CPZ = \frac{0}{a}$: eodemque caſu punctum P propius propiusque ad centrum accedat, & cum eo tandem coincidat; ita fit ſin. $CPZ = \frac{0}{0}$: & cum hoc caſu unica ſit conditio, nempe ut recta ducenda per unum tantum punctum datum transeat; ſignum hoc indicat indeterminationem poſitionis lineæ magnitudine datæ. Fig. 42.

Obſervatio. Si fuiſſet tantum $a = 0$, non autem $b = r$: foret ſin. $CPZ = \frac{\sqrt{(rr-bb)}}{0}$; & hoc ſymbolum eſſet ſignum impoſſibilitatis (Cap. IX.).

Alterum exemplum. Sint duo puncta poſitione data; ſit & tertium punctum poſitione datum, per quod ducenda ſit recta talis, ut perpendicula ex duobus prioribus punctis datis in eam demiſſa ſint inter ſe in ratione data. Fig. 43.

Sint A, B duo priora puncta data; ſit P tertium punctum datum: & quæratur recta PX, in quam demiſſa perpendicula Aa, Bb ſint inter ſe in ratione data. Agatur recta AB, & dividatur in C in ratione data: recta PC eſt recta quæſita. Ut problema ſit determinatum; neceſſe eſt, ut P & C puncta ſint diverſa: ſi ſecus eveniat, unicum datur punctum, per quod recta ducenda ſit, & proinde poſitio ejus eſt indeterminata; quod calculo etiam indicatur.

Sit $AC = a$, $PC = b$, $AP = c$; in triangulo APC fit ſin. $C =$

$= \frac{2\sqrt{\left(\frac{a+b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{-a+b+c}{2}\right)}}{ab}$. Ratione data manente eadem, ſeu eadem manente AC, evaneſcat PC: fiet $AP = AC$, ſeu $a = c$, & $b = 0$; unde ſin. $C = 2\frac{0}{0}$: quod ſignum indicat, omnes rectas per C ductas conditioni propoſitæ ſatisfacere.

Hæc poſſent ad numerum quemcunque punctorum poſitione datorum ap-

plicari; & huc quoque revocatur indeterminatio, cui anſam præbet problema locale, quod expoſui in opuſculo meo inſcripto: *Polygonometrie*, pag. 72—93.

Indeterminatio hæc, ubi occurrit, affectiones quasdam generales quantitatum, quæ locum ei præbent, poteſt indicare. Sic v. gr. poſtremum exemplum intime connectitur cum proprietate centri gravitatis, de qua vide §. 125 ſq. Sufficiat paucis exemplis elementaribus aſſertum hoc iterum illuſtrare.

Circulo dato inſcribendum ſit quadrilaterum, cujus anguli dantur.

Fig. 44. Sit $ABA'B'$ quadrilaterum propoſitum, inſcribendum circulo, cujus centrum C & radius CA. Agantur radii CA, CB, CA', CB'.

Sit $CAB = CBA = x$

Erunt $CBA' = CA'B = B - x$

$CA'B' = CB'A' = A' - B + x$

$CB'A = CAB' = B' - A' + B - x = A - x$.

Unde angulus x fit indeterminatus: & ſimul $A + A' = B + B'$; ſeu diſcimus, ſummas angulorum alterne ſumtorum eſſe invicem æquales. Idem valet de quavis figura rectilinea numeri laterum paris, circulo inſcribenda. Huc etiam pertinet caſus indeterminatus problematis in Geodæſia tantopere utilis; quo, tribus punctis poſitione datis, quæritur quartum punctum, obſervatis ex hoc puncto angulis, ſub quibus mutuæ priorum punctorum diſtantiæ apparent.

Idem dicatur de ſummis laterum alterne ſumtorum figurarum circulo circumſcriptarum.

Sint A, B, C latera alicujus trianguli;

a, b, c anguli his lateribus oppoſiti: quamdiu latera A & B ſunt invicem inæqualia, ac proinde etiam anguli a & b invicem inæquales; eſt $\text{tang.}\frac{a+b}{2} : \text{tang.}\frac{a-b}{2} = A+B : A-B$: data igitur ratione (inæqualitatis) laterum A & B, & data differentia angulorum a & b, determinatur tangens dimidiæ ſummæ horum angulorum per æquationem $\text{tang.}\frac{a+b}{2} = \text{tang.}\frac{a-b}{2} \times \frac{A+B}{A-B}$.

Sint autem latera A, B invicem æqualia: fit $\text{tang.}\frac{a+b}{2} = \text{tang.}\frac{a-b}{2} \times \frac{1}{0}$; & hoc foret ſignum impoſſibilitatis, niſi eſſet ſimul $a = b$, & $\text{tang.}\frac{a-b}{2} = 0$: unde

tang.

tang.$\frac{a+b}{2} = \frac{0}{0}$. Signum ideo indeterminationis $\frac{0}{0}$ nos monet: quod, ſi nulla ſit laterum differentia, etiam nulla ſit angulorum oppoſitorum differentia.

Idem dicatur de altera propoſitione, quæ unum eſt etiam ex præcipuis trigonometriæ planæ fundamentis. Sit B baſis alicujus trianguli, in quam ex vertice oppoſito b agatur recta perpendicularis; & ſint A', C' ſegmenta baſis lateribus A & C adjacentia. Quamdiu A & C ſunt invicem inæqualia, ſtat proportio $A+C : A-C = B : A'-C'$; unde $A+C = B \times \frac{A-C}{A'-C'}$: ſi vero $A'=C'$, & proinde $A=C$; fit $A+C = B \times \frac{0}{0}$, quæ eſt expreſſio indeterminata. Numerus nempe triangulorum æquicrurorum ſuper data baſi conſtruendorum eſt illimitatus.

Sed hæc ſufficiant de re ad elementa pertinentia: nec immoror oſtendendo nexui inter propoſitiones, ab antiquis Poriſmata nuncupatas, & ſignum $\frac{0}{0}$ caſu, quo indicat indeterminationem.

Caput decimum septimum.

De theoremate Tayloriano ad functiones duarum pluriumve variabilium extenſo; et de rationibus differentialibus atque integralibus earundem functionum.

§. 160.

Sit P functio duarum quantitatum mutabilium x, y. Valores, quos functio hæc recipit, quando quantitas x ſola, aut quantitas y ſola, mutationes Δx, Δy reſpective patiuntur, denotentur ſignis ${}^{x}P$, ${}^{y}P$. Valor, quem eadem functio recipit, quando x & y mutationes Δx & Δy ſucceſſive patiuntur, ponatur $\overset{x}{{}^{y}P}$; pariterque valor, quem functio hæc recipit, quando y & x mutationes Δy & Δx ſucceſſive patiuntur, deſignetur per $\overset{y}{{}^{x}P}$.

Functio $\overset{x}{{}^{y}P}$ ex functione ${}^{x}P$ oritur, ſi in ea $y+\Delta y$ loco y ſubſtituitur; pariterque functio $\overset{y}{{}^{x}P}$ ex functione ${}^{y}P$ oritur, ſi in ea $x+\Delta x$ loco x ſubſtituitur. Proinde duæ expreſſiones $\overset{x}{{}^{y}P}$, $\overset{y}{{}^{x}P}$ unum eundemque functionis P valorem deſi-

 gnant:

gnant: eum nempe, quem obtinet, ſi quantitatum x & y loco quantitates $x+\Delta x$, $y+\Delta y$ ſimul ſubſtituantur.

Quoniam notatio a mathematicis uſurpata, qua exponentes differentiales functionum duarum pluriumve quantitatum mutabilium deſignant, minus commoda mihi videtur; liceat aliam proponere, calculo (mea quidem ſententia) aptiorem. Scilicet exponens differentialis functionis P duarum pluriumve variabilium, quatenus x ſola mutatur, denotetur ſigno ${}^{x}d'P$; & exponentes differentiales ſucceſſivi eodem modo ſumti ponantur ${}^{x}d''P$, ${}^{x}d'''P$, ${}^{x}d^{iv}P$ .... Tum exponens differentialis functionis ${}^{x}d'P$, quatenus y ſola mutatur, deſignetur ${}^{y}d'\,{}^{x}d'P$; & generatim exponens differentialis m^{ti} ordinis functionis ${}^{x}d^{N}P$, quatenus y ſola in hac functione mutatur, ſit ${}^{y}d^{M}\,{}^{x}d^{N}P$.

Quibus ſuppoſitis, paucis oſtendam: quomodo theorema Taylorianum, de functionibus unius tantum quantitatis mutabilis demonſtratum, ad functiones duarum pluriumve quantitatum mutabilium extendatur; a functionibus duarum quantitatum mutabilium x, y, quæ fiant $x+\Delta x$, $y+\Delta y$, ordiundo.

§. 161. Per theorema Taylorianum (Cap. III.) eſt

$${}^{x}P = P+\frac{\Delta x}{1}{}^{x}d'P+\frac{\Delta x^2}{1.2}{}^{x}d''P+\frac{\Delta x^3}{1.2.3}{}^{x}d'''P+\frac{\Delta x^4}{1\ldots4}{}^{x}d^{iv}P+\frac{\Delta x^5}{1\ldots5}{}^{x}d^{v}P+\ldots.$$

$${}^{y}P = P+\frac{\Delta y}{1}{}^{y}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''P+\frac{\Delta y^4}{1\ldots4}{}^{y}d^{iv}P+\frac{\Delta y^5}{1\ldots5}{}^{y}d^{v}P+\ldots.$$

Proinde

$$\begin{aligned}
{}^{x}_{y}P = P &+\frac{\Delta y}{1}{}^{y}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''P+\frac{\Delta y^4}{1\ldots4}{}^{y}d^{iv}P+\frac{\Delta y^5}{1\ldots5}{}^{y}d^{v}P+\ldots. \\
&+\frac{\Delta x}{1}\Big({}^{x}d'P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d'P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d'P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''\,{}^{x}d'F+\frac{\Delta y^4}{1\ldots4}{}^{y}d^{iv}\,{}^{x}d'P+\ldots\Big) \\
&+\frac{\Delta x^2}{1.2}\Big(\ldots\ldots\ {}^{x}d''P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d''P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d''P+\frac{\Delta y^3}{1.2.3}{}^{y}d'''\,{}^{x}d''P+\ldots\Big) \\
&+\frac{\Delta x^3}{1.2.3}\Big(\ldots\ldots\ldots\ {}^{x}d'''P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d'''P+\frac{\Delta y^2}{1.2}{}^{y}d''\,{}^{x}d'''P+\ldots\Big) \\
&+\frac{\Delta x^4}{1\ldots4}\Big(\ldots\ldots\ldots\ldots\ {}^{x}d^{iv}P+\frac{\Delta y}{1}{}^{y}d'\,{}^{x}d^{iv}P+\ldots\Big) \\
&+\frac{\Delta x^5}{1\ldots5}\Big(\ldots\ldots\ldots\ldots\ldots\ {}^{x}d^{v}P+\ldots.\Big)
\end{aligned}$$

Pari-

Pariter

$$\overset{y}{{}^{x}}P = P + \frac{\Delta x}{1}{}^{x}d'P + \frac{\Delta x^2}{1.2}{}^{x}d''P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''P + \frac{\Delta x^4}{1...4}{}^{x}d^{iv}P + \frac{\Delta x^5}{1...5}{}^{x}d\;P + \dots$$

$$+ \frac{\Delta y}{1}\Big({}^{y}d'P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d'P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d'P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''\,{}^{y}d'P + \frac{\Delta x^4}{1..4}{}^{x}d^{iv}\,{}^{y}d'P + \dots\Big)$$

$$+ \frac{\Delta y^2}{1.2}\Big(\;.\;\;.\;\;.\;\;{}^{y}d''P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d''P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d''P + \frac{\Delta x^3}{1.2.3}{}^{x}d'''\,{}^{y}d''P + \dots\Big)$$

$$+ \frac{\Delta y^3}{1.2.3}\Big(\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;{}^{y}d'''P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d'''P + \frac{\Delta x^2}{1.2}{}^{x}d''\,{}^{y}d'''P + \dots\Big)$$

$$+ \frac{\Delta y^4}{1...4}\Big(\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;{}^{y}d^{iv}P + \frac{\Delta x}{1}{}^{x}d'\,{}^{y}d^{iv}P + \dots\Big)$$

$$+ \frac{\Delta y^5}{1...5}\Big(\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;{}^{y}d^{v}P + \dots\Big)$$

Atqui $\overset{x}{{}^{y}}P = \overset{y}{{}^{x}}P$; proinde æquando terminos harum expreſſionum mutationibus Δx & Δy eodem modo affectos, ſequentes oriuntur æquationes:

${}^{y}d'\,{}^{x}d'P = {}^{x}d'\,{}^{y}d'P$

${}^{y}d''\,{}^{x}d'P = {}^{x}d'\,{}^{y}d''P \qquad {}^{y}d'\,{}^{x}d''P = {}^{x}d''\,{}^{y}d'P$

${}^{y}d'''\,{}^{x}d'P = {}^{x}d'\,{}^{y}d'''P \qquad {}^{y}d''\,{}^{x}d''P = {}^{x}d'\,{}^{y}d''P \qquad {}^{y}d'\,{}^{x}d'''P = {}^{x}d'''\,{}^{y}d'P$

${}^{y}d^{iv}\,{}^{x}d'P = {}^{x}d'\,{}^{y}d^{iv}P \qquad {}^{y}d''\,{}^{x}d'''P = {}^{x}d'''\,{}^{y}d''P \qquad {}^{y}d''\,{}^{x}d'''P = {}^{x}d'''\,{}^{y}d''P \qquad {}^{y}d'\,{}^{x}d^{iv}P = {}^{x}d^{iv}\,{}^{y}d'P$

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Generatim ${}^{y}d^{N}\,{}^{x}d^{M}P = {}^{x}d^{M}\,{}^{y}d^{N}P$.

Exempla. 1°. Sit $P = xy$. ${}^{x}d'P = y \qquad {}^{y}d'P = x$

${}^{x}d''P = 0 \qquad {}^{y}d''P = 0$

${}^{y}d'\,{}^{x}d'P = 1 \qquad {}^{x}d'\,{}^{y}d'P = 1.$

2°. Sit $P = x^4y^3$. ${}^{x}d'P = 4x^3y^3 \qquad {}^{y}d'P = 3x^4y^2 \qquad {}^{y}d'\,{}^{x}d'P = 4.3.x^3yy = {}^{x}d'\,{}^{y}d'P$

${}^{x}d''P = 4.3xxy^3 \qquad {}^{y}d''P = 3.2x^4y \qquad {}^{y}d''\,{}^{x}d'P = 4.3.2x^3y = {}^{x}d'\,{}^{y}d''P$

${}^{x}d'''P = 4.3.2xy^3 \qquad {}^{y}d'''P = 3.2.1.x^4 \qquad {}^{y}d'''\,{}^{x}d'P = 4.3.2.1x^3 = {}^{x}d'\,{}^{y}d'''P$

${}^{x}d^{iv}P = 4.3.2.1.y^3$

§. 162.

§. 162. Ex §. præcedente (inſiſtendo veſtigiis Capitum I. II.) deducitur determinatio mutationum ſimultanearum duarum quantitatum variabilium x, y, atque functionis ipſarum P; ac ſpeciatim exponentis differentialis harum mutationum. Conſideratis nimirum functione P & quantitatibus mutabilibus x, y, tanquam functionibus unius ejusdemque quantitatis mutabilis p;

erit $\frac{dP}{dp} = \frac{dx}{dp}{}^{x}d'P + \frac{dy}{dp}{}^{y}d'P$.

Exempla. 1°. Sit $P = xy$. Erit $\frac{dP}{dp} = \frac{dx}{dp}y + \frac{dy}{dp}x$ (ut notum §. 26.).

2°. Sit $P = x^4y^3$; itaque ${}^{x}d'P = 4x^3y^3$, ${}^{y}d'P = 3x^4yy$.

$$\text{Erit } \frac{dP}{dp} = 4x^3y^3\frac{dx}{dp} + 3x^4yy\frac{dy}{dp}.$$

Scilicet ut habeatur exponens differentialis $\frac{dP}{dp}$: ſumantur exponentes differentiales ${}^{x}d'P$, ${}^{y}d'P$; qui multiplicentur per exponentes $\frac{dx}{dp}$, $\frac{dy}{dp}$ reſpective.

§. 163. Data igitur æquatione differentiali $\frac{dP}{dx} = \frac{dx}{dp}X + \frac{dy}{dp}Y$: ut æquationi huic reſpondeat æquatio integralis, terminis finitis expreſſa; oportet, ſit $X = {}^{x}d'P$, & $Y = {}^{y}d'P$, proindeque ${}^{y}d'X = {}^{x}d'Y$.

Exempla. 1°. Sit $\frac{dP}{dp} = \frac{dx}{dp}x + \frac{dy}{dp}$; igitur $X = x$, $Y = y$: cum ita ſit ${}^{y}d'X = 0$, ${}^{x}d'Y = 0$; contradictionem non involvit, ut æquationi differentiali propoſitæ reſpondeat æquatio integralis, quæ ſit $2P = xx + yy$.

$$2°.\ \text{Sit } \frac{dP}{dp} = \frac{dx}{dp}(3xxyy + y^4) + \frac{dy}{dp}(2x^3y + 4xy^3 + 5y^4)$$

$$= \frac{dx}{dp}X \qquad + \frac{dy}{dp}Y.$$

Tum ${}^{y}d'X = 2.3xxy + 4y^3$, ${}^{x}d'Y = 2.3xxy + 4y^3$;

ac ſit $P = x^3yy + xy^4 + y^5$.

3°. Sit

3°. Sit $\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{\mathrm{d}x}{\mathrm{d}p}(y^4 + x^3 y) + \frac{\mathrm{d}y}{\mathrm{d}p}(xxyy + y^4)$

$= \frac{\mathrm{d}x}{\mathrm{d}p} X \qquad + \frac{\mathrm{d}y}{\mathrm{d}p} \Upsilon.$

Fiunt ${}^{y}\mathrm{d}'X = 4y^3 + x^3$, ${}^{x}\mathrm{d}'\Upsilon = 2xyy$, et æquationi propoſitæ nulla reſpondet æquatio integralis terminis finitis expreſſa.

§. 164. Ex æquatione differentiali primi gradus fluunt æquationes differentiales graduum reliquorum: quod æquationis differentio-differentialis exemplo illuſtrare ſufficiat.

Quoniam $\frac{\mathrm{d}P}{\mathrm{d}p} = \frac{\mathrm{d}x}{\mathrm{d}p}\,{}^{x}\mathrm{d}'P + \frac{\mathrm{d}y}{\mathrm{d}p}\,{}^{y}\mathrm{d}'P;$

ſequitur $\frac{\mathrm{dd}P}{\mathrm{d}p^2} = \frac{\mathrm{dd}x}{\mathrm{d}p^2}\,{}^{x}\mathrm{d}'P + \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^{y}\mathrm{d}''P + \frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P$

$+ \frac{\mathrm{d}y}{\mathrm{d}p}\cdot\frac{\mathrm{d}x}{\mathrm{d}p}\,{}^{x}\mathrm{d}'\,{}^{y}\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^{y}\mathrm{d}''P + \frac{\mathrm{dd}y}{\mathrm{d}p^2}\,{}^{y}\mathrm{d}''P$

$= \frac{\mathrm{dd}x}{\mathrm{d}p^2}\,{}^{x}\mathrm{d}'P + \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^{x}\mathrm{d}''P + 2\frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^{y}\mathrm{d}''P + \frac{\mathrm{dd}y}{\mathrm{d}p^2}\,{}^{y}\mathrm{d}''P$

$= \qquad \left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 {}^{x}\mathrm{d}''P + 2\frac{\mathrm{d}x}{\mathrm{d}p}\cdot\frac{\mathrm{d}y}{\mathrm{d}p}\,{}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 {}^{y}\mathrm{d}''P$, ſi exponentes differentiales $\frac{\mathrm{d}x}{\mathrm{d}p}$, $\frac{\mathrm{d}y}{\mathrm{d}p}$ ſupponuntur eſſe conſtàntes.

§. 165. Ex formula pro duabus quantitatibus mutabilibus §. 161. expoſita, ſequitur formula pro functione trium quantitatum mutabilium. Fit enim

$${}^{\substack{x\\y\\z}}P = P + \frac{\Delta z}{1}\,{}^{z}\mathrm{d}'P + \frac{\Delta z^2}{1.2}\,{}^{z}\mathrm{d}''P + \frac{\Delta z^3}{1.2.3}\,{}^{z}\mathrm{d}'''P + \frac{\Delta z^4}{1 \ldots 4}\,{}^{z}\mathrm{d}^{\mathrm{iv}}P + \ldots.$$

$$+ \frac{\Delta x}{1}\left({}^{x}\mathrm{d}'P + \frac{\Delta z}{1}\,{}^{z}\mathrm{d}'\,{}^{x}\mathrm{d}'P + \frac{\Delta z^2}{1.2}\,{}^{z}\mathrm{d}''\,{}^{x}\mathrm{d}'P + \frac{\Delta z^3}{1.2.3}\,{}^{z}\mathrm{d}'''\,{}^{x}\mathrm{d}'P + \ldots.\right)$$

$$+ \frac{\Delta y}{1}\left({}^{y}\mathrm{d}'P + \frac{\Delta z}{1}\,{}^{z}\mathrm{d}'\,{}^{y}\mathrm{d}'P + \frac{\Delta z^2}{1.2}\,{}^{z}\mathrm{d}''\,{}^{y}\mathrm{d}'P + \frac{\Delta z^3}{1.2.3}\,{}^{z}\mathrm{d}'''\,{}^{y}\mathrm{d}'P + \ldots.\right)$$

$$+ \frac{\Delta x^2}{1.2}\left({}^{x}\mathrm{d}''P + \frac{\Delta z}{1}\,{}^{z}\mathrm{d}'\,{}^{x}\mathrm{d}''P + \frac{\Delta z^2}{1.2}\,{}^{z}\mathrm{d}''\,{}^{x}\mathrm{d}''P + \ldots.\right)$$

$$+ 2\frac{\Delta x\,\Delta y}{1.2}\left({}^{x}\mathrm{d}'\,{}^{y}\mathrm{d}'P + \frac{\Delta z}{1}\,{}^{z}\mathrm{d}'\,{}^{x}\mathrm{d}'\,{}^{y}\mathrm{d}'P + \frac{\Delta z^2}{1.2}\,{}^{z}\mathrm{d}''\,{}^{x}\mathrm{d}'\,{}^{y}\mathrm{d}'P + \ldots.\right)$$

$$+\frac{\Delta y^2}{1.2}\Big(\qquad\qquad\qquad {}^{y}d''p \quad + \frac{\Delta z}{1}\,{}^{z}d'\,{}^{y}d''p + \ldots .\Big)$$

$$+\frac{\Delta x^3}{1.2.3}\Big(\qquad\qquad\qquad\qquad {}^{x}d'''p \;+ \ldots .\Big)$$

$$+3\frac{\Delta x^2.\Delta y}{1.2.3}\Big(\qquad\qquad\qquad {}^{x}d''\,{}^{y}d'p + \ldots .\Big)$$

$$+3\frac{\Delta x.\Delta y^2}{1.2.3}\Big(\qquad\qquad\qquad {}^{x}d'\,{}^{y}d''p + \ldots .\Big)$$

$$+\frac{\Delta y^3}{1.2.3}\Big(\qquad\qquad\qquad\qquad {}^{y}d'''p + \ldots .\Big)$$

$$+ \;-\;\;-\;\;-\;\;-\;\;-\;\;-$$

$$+ \;-\;\;-\;\;-\;\;-\;\;-\;\;-$$

Si varietur ordo, juxta quem tres quantitates mutabiles x, y, z ſucceſſive poſſunt variari; ſex nempe modis, quibus inter ſe permutantur; expreſſiones $\overset{x}{\overset{y}{{}^{z}P}}$, $\overset{x}{\overset{z}{{}^{y}P}}$, $\overset{y}{\overset{x}{{}^{z}P}}$, $\overset{y}{\overset{z}{{}^{x}P}}$, $\overset{z}{\overset{x}{{}^{y}P}}$, $\overset{z}{\overset{y}{{}^{x}P}}$ erunt inter ſe æquales. Unde varia conſequuntur theoremata; & nominatim æqualitas terminorum, qui iisdem productis mutationum Δx, Δy, Δz afficiuntur. Sic v. gr. obtinentur æquationes ſequentes: ${}^{z}d'\,{}^{x}d'\,{}^{y}d'p = {}^{z}d'\,{}^{y}d'\,{}^{x}d'p = {}^{x}d'\,{}^{y}d'\,{}^{z}d'p = {}^{x}d'\,{}^{z}d'\,{}^{y}d'p = {}^{y}d'\,{}^{x}d'\,{}^{z}d'p = {}^{y}d'\,{}^{z}d'\,{}^{x}d'p$. Ex quibus porro nectuntur corollaria analoga iis, quæ de functionibus duarum variabilium notavimus §. 162. 163.

Hinc facilis eſt tranſitus ad functiones quatuor, quinque, & plurium quotcunque quantitatum mutabilium. Quare viam indicaſſe ſufficiat, qua tractatio hæc ab idea infiniti liberatur; & quod ad applicationes attinet, qui plura deſideraverit, adeat Euleri *Calculum differentialem* Cap. VII. & VIII. partis prioris.

Caput decimum octavum.

De maximis et minimis; et de punctis flexus contrarii curvarum.

Quæſtiones ad maxima & minima pertinentes adeo frequenter occurrunt, non in pura tantum matheſi, ſed potiſſimum quoque in applicata, quæ uberrimis utiliſſimisque earum applicationibus anſam præbet; ut caput hoc ſedulo evol-

evolvi mereatur. Quoniam autem intimus eſt ipſarum cum quibusdam curvarum ſymptomatibus nexus, quorum contemplatio magnam huic doctrinæ lucem affundit; utrumque argumentum ſimul complecti e re eſſe cenſeo.

§. 166. *Definitio* 1. Sit P functio quantitatis alicujus mutabilis x, quæ facta $x = a$ $\begin{matrix}\text{major}\\ \text{minor}\end{matrix}$ fiat, quam ſi quantitati x major minorve valor $x + \Delta x$, $x - \Delta x$ tribuatur, utut parum valores hi ab a differant: valor functionis P valori $x = a$ reſpondens vocatur $\begin{matrix}\textit{maximus}\\ \textit{minimus}\end{matrix}$.

Exempla. Numerus datus a in duas partes x, $x - a$ dividatur; & fiat productum $x(a - x)$ ex his partibus. Productum hoc functio eſt unius harum partium v. gr. x; qua creſcente a zero inde usque ad $\frac{1}{2}a$ creſcit etiam productum $x(a - x)$. Eadem autem parte creſcere pergente, idem productum decreſcit: ideoque productum hoc eſt omnium maximum, quando pars x æqualis eſt dimidio numero dividendo; ſeu quando ambæ partes ſunt invicem æquales.

Contra ſi fiant quadrata ex iisdem partibus; ſumma horum eſt omnium minima, quando ambæ partes ſunt invicem æquales.

Definitio 2. Sit M punctum quodpiam alicujus curvæ: ex quo ducantur recta curvam tangens TMT'; & recta MP axi ordinatim applicata, cui reſpondet abſciſſa AP. Tum $\begin{matrix}\text{aucta}\\ \text{imminuta}\end{matrix}$ abſciſſa, quæ fit $\begin{matrix}AP'\\ A'P\end{matrix}$, agantur $\begin{matrix}P'M'\\ 'P'M\end{matrix}$ rectæ axi ordinatim applicatæ, quæ curvæ in $\begin{matrix}M'\\ 'M\end{matrix}$ punctis, & tangenti in $\begin{matrix}N'\\ 'N\end{matrix}$ punctis occurrant. Si ſit $N'P' \gtrless M'P'$, utut parum abſciſſa AP' ab abſciſſa AP differat; arcus MM' dicitur $\begin{matrix}\textit{concavus}\\ \textit{convexus}\end{matrix}$ verſus axem AB: & ſi ſimul ſit $'N'P \gtrless {'M'P}$, totus arcus $'MMM$ verſus eundem axem eſt $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$. Si vero ſimul ſit $N'P' \gtrless M'P'$ & $'N'P \lessgtr {'M'P}$, quo caſu arcus MM' $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$ eſt verſus axem AB, dum arcus $'MM$ eſt verſus eundem axem $\begin{matrix}\text{convexus}\\ \text{concavus}\end{matrix}$; tum M punctum ſeparat arcum $\begin{matrix}\text{concavum}\\ \text{convexum}\end{matrix}$ curvæ ab ejus arcu $\begin{matrix}\text{convexo}\\ \text{concavo}\end{matrix}$, & dicitur *punctum flexus contrarii* curvæ.

Fig. 45. 1°. 2°.

Fig. 46.

§. 161. Hinc jam oſtendi poteſt nexus, qui ſymptomata inter curvarum, quibus verſus axem $\begin{matrix}\text{concavæ}\\ \text{convexæ}\end{matrix}$ ſunt, & doctrinam functionum, quæ $\begin{matrix}\text{maximæ}\\ \text{minimæ}\end{matrix}$ fiunt, intercedit.

Etenim

Etenim quando arcus totus eſt verſus axem, ad quem refertur, $\genfrac{}{}{0pt}{}{\text{concavus}}{\text{convexus}}$: ſi recta duci poteſt arcum hunc tangens, eademque axi parallela; recta axi ordinatim applicata ex puncto contactus ducta $\genfrac{}{}{0pt}{}{\text{major}}{\text{minor}}$ eſt rectis axi ordinatim applicatis ad utramque ejus partem ſitis; proinde ordinata hæc omnium $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ eſt. Quare ordinatis curvæ ſumtis proportionalibus functioni P quantitatis mutabilis, quæ ipſa per abſciſſas AP deſignetur; functio P omnium $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ determinatur per rectam axi ordinatim applicatam a puncto curvæ, ubi hanc tangit recta axi parallela.

Quoniam autem, quando tangens axi eſt parallela, fit $\frac{dy}{dx}=0$, ſeu $\frac{dP}{dx}=0$ (§. 93.); $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ determinantur, ſi fiat $\frac{dP}{dx}=0$.

§. 168. Si curva propoſita, cujus ordinatæ ſunt functioni P proportionales, unico conſtet ramo, qui totus ſit verſus axem $\genfrac{}{}{0pt}{}{\text{concavus}}{\text{convexus}}$: recta tangens axi parallela (ſi qua ſit) eſt unica; & æquationis $\frac{dP}{dx}=0$ unica eſt radix realis. Si vero curva propoſita ſit undulatoria, ita ut alternis vicibus concava & convexa fiat verſus axem, ad quem refertur: in unaquaque unda agi poterit recta tangens axi parallela; & proinde totidem erunt ordinatæ alternis vicibus maximæ aut minimæ, quot ſunt undæ curvæ propoſitæ. Multitudo hæc $\genfrac{}{}{0pt}{}{\text{maximorum}}{\text{minimorum}}$ deſignatur multitudine radium realium æquationis $\frac{dP}{dx}=0$, quibus etiam $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ functionis P valores alternis vicibus reſpondent.

§. 169. In definitionibus præcedentibus (§. 166.) ſuppoſui: valorem a quantitatis mutabilis x, cui $\genfrac{}{}{0pt}{}{\text{maximus}}{\text{minimus}}$ functionis P valor reſpondet, ejusmodi eſſe, ut ſumi poſſint quantitatis x valores ipſa a tam majores quam minores; ſeu (quod eodem redit) ſuppoſui: curvam, cujus ordinatæ ſunt functioni P proportionales, ad utramque ordinatæ omnium $\genfrac{}{}{0pt}{}{\text{maximæ}}{\text{minimæ}}$ partem extendi. Fieri autem poteſt, ut valor a quantitatis mutabilis x, cui $\genfrac{}{}{0pt}{}{\text{maximus}}{\text{minimus}}$ functionis ejus valor reſpondet, ſit ſimul & ipſe $\genfrac{}{}{0pt}{}{\text{maximus}}{\text{minimus}}$ quantitatis mutabilis x valor: ita ut

non

non poſſint ſimul accipi valores $x+\Delta x$, $x-\Delta x$, ſeu abſciſſæ curvæ, his valoribus reſpondentes; & curva ceſſet ad alteram ordinatæ MP, abſciſſæ $x=a$ reſpondentis, partem. Quod ſi locum habeat: ordinatæ abſciſſis $x+\Delta x$ v. gr. reſpondentes fiunt imaginariæ; & proinde functio P radice $(x-a)^{\frac{2m+1}{2n}}$ vel $(a-x)^{\frac{2m+1}{2n}}$ afficitur.

Ut argumentum hoc, quantum fieri poteſt, dilucide pertractem; ſingulare hoc $\genfrac{}{}{0pt}{}{\text{maximorum}}{\text{minimorum}}$ genus ſeorſim perpendam: & de iis functionibus, quæ nonniſi poteſtates $(x-a)^n$, $(a-x)^n$, quarum exponens n eſt numerus integer poſitivus, involvunt, unice primum agam; ceteraſque aliis potentiis affectas demum conſiderabo, poſtquam, quæ ad priores pertinent, erunt declarata.

Et cum adeo arctus ſit nexus inter ſymptomata curvarum, quibus ſunt verſus axem $\genfrac{}{}{0pt}{}{\text{concavæ}}{\text{convexæ}}$, & ſymptomata functionum, quibus $\genfrac{}{}{0pt}{}{\text{maximæ}}{\text{minimæ}}$ fieri poſſunt; priorem determinationem præmittam.

§. 170. Sit M punctum curvæ, ad quod ducta eſt recta tangens TT'. Sit M' aliud curvæ hujus punctum, puncto M utut proximum; a quo agatur recta $M'P'$ axi ordinatim applicata, quæ tangenti TT' in N' puncto occurrat. Tum ex puncto M agatur Mm' recta axi parallela, quæ rectæ $M'P'$ in m' puncto occurrat. Ad alteram ordinatæ MP partem agatur quoque recta $'M'P$, axi ordinatim applicata, quæ tangenti in $'N$ puncto, & rectæ Mm' in $'m$ puncto occurrat.

Sint $MP=y$, $M'P'=y'$, $'M'P='y$, $PP'='PP=\Delta x$.

Erit $$y'=y+\frac{\Delta x}{1}\cdot\frac{dy}{1}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}+\ldots \quad (\S.\ 32.)$$

$$'y=y-\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}-\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}-\ldots$$

ſeu $$\genfrac{}{}{0pt}{}{y'}{'y}=y\pm\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}\pm\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}\pm\ldots$$

Atqui $\genfrac{}{}{0pt}{}{N'P'}{'N'P}=y\pm\frac{\Delta x}{1}\cdot\frac{dy}{dx}$ (§. 40.)

 Ergo

Ergo caſu concavitatis $\frac{N'M'}{'N'M} = -\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1...4} \cdot \frac{d^4y}{dx^4} \mp \ldots$

caſu convexitatis $\frac{N'M'}{'N'M} = +\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta y^4}{1...4} \cdot \frac{d^4y}{dx^4} \pm \ldots$

Poſito autem, functionem p alias factoris $x-a$, ipſi $x=a$ reſpondentis, poteſtates non continere, niſi quarum exponentes ſunt numeri integri & poſitivi: exponentes differentiales ſucceſſivi $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, ... aut factore $x-a$ non afficiuntur, aut non alias involvunt ejus poteſtates, niſi quarum exponentes ſunt integri poſitivi; proinde facto $x=a$, exponentes hi differentiales aut finitam obtinent magnitudinem, aut evaneſcunt, neque impoſſibilis aut infiniti $\frac{1}{(x-a)^n}$ ſigno afficiuntur.

Quo poſito, ut curva ſit ad utramque puncti M partem concava; oportet, ſit $-\left(\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1...4} \cdot \frac{d^4y}{dx^4} \pm \ldots\right)$ quantitas poſitiva, útut parvus ſit ipſius Δx valor: proinde ſi $\frac{ddy}{dx^2}$ non evaneſcat; oportet, ſit $\frac{ddy}{dx^2}$ quantitas negativa (§. 16.) propter Δx^2 ſemper poſitivum.

Caſu autem convexitatis neceſſe eſt, ut ſit

$+\left(\frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1...4} \cdot \frac{d^4y}{dx^4} \pm \ldots\right)$ quantitas poſitiva, utcunque exiguus ſit ipſius Δx valor; quod (propter Δx^2 ſemper poſitivum) fieri nequit, niſi ſit $\frac{ddy}{dx^2}$ quantitas poſitiva.

Proinde, exponente differentiali $\frac{ddy}{dx^2}$ non evaneſcente, curva verſus axem $\begin{matrix}\text{concava}\\\text{convexa}\end{matrix}$ eſt, prouti exponens differentialis $\frac{ddy}{dx^2}$ eſt $\begin{matrix}\text{negativus}\\\text{poſitivus}\end{matrix}$.

§. 171. Quoniam autem $\overset{'}{\underset{,}{y}} = y \pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1..4} \cdot \frac{d^4y}{dx^4} \pm \ldots$; caſu, quo y eſt maxima, debet eſſe ſimul

$\mp \frac{\Delta x}{1} \cdot \frac{dy}{dx} - \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1...4} \cdot \frac{d^4y}{dx^4} \mp \ldots > 0$: quod juxta ſuppoſitionem

nem (§. 169.) fieri nequit, niſi ſit $\frac{dy}{dx} = 0$, & $\frac{ddy}{dx^2}$ negativa, ſeu curva verſus axem concava.

Caſu vero, quo y minima eſt, debet eſſe
$\pm \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{1.2} \cdot \frac{ddy}{dx^2} \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3 y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4 y}{dx^4} \pm \ldots > 0$; quod fieri nequit (juxta eaſdem ſuppoſitiones), niſi ſit $\frac{dy}{dx} = 0$, & $\frac{ddy}{dx^2}$ poſitiva, ſeu curva verſus axem convexa.

Poſita igitur functione P hujus formæ $\frac{\varphi x (x-a)^n + \varphi' x}{\varphi x (a-x)^n + \varphi' x}$, ubi n eſt numerus integer poſitivus: facto $\frac{dP}{dx} = 0$, cui reſpondeat $x = a$; exponentibus differentialibus ſucceſſivis $\frac{ddP}{dx^2}, \frac{d^3 P}{dx^3}, \frac{d^4 P}{dx^4}, \ldots$ factorem impoſſibilem $\frac{1}{(x-a)^n}$ non involventibus, & exponente differentiali $\frac{ddy}{dx^2}$ non evaneſcente; functio P fit $\begin{smallmatrix}\text{maxima}\\ \text{minima}\end{smallmatrix}$, prouti exponens differentialis $\frac{ddP}{dx^2}$, ſubſtitutione $x = a$ in ejus expreſſione facta, eſt $\begin{smallmatrix}\text{negativus}\\ \text{poſitivus}\end{smallmatrix}$.

Exemplum primum. Sit $P = 2ax - xx$; ideoque $\frac{dP}{dx} = 2a - 2x$, $\frac{ddP}{dx^2} = -2$: facto $\frac{dP}{dx} = 0$, eſt $x = a$, & $\frac{ddP}{dx^2}$ eſt ſemper negativus; proinde valori $x = a$ reſpondet P maximum.

Exemplum ſecundum. Sit $P = xx + (a-x)^2$; igitur $\frac{dP}{dx} = 2(2x-a)$, $\frac{ddP}{dx^2} = +4$: facto $\frac{dP}{dx} = 0$, eſt $x = \frac{1}{2}a$, & $\frac{ddP}{dx^2}$ ſemper eſt poſitivus; proinde functio P, ipſi $x = \frac{1}{2}a$ reſpondens, eſt minima.

Exemplum tertium. Sit $P = xx - (a-x)^2$: erit $\frac{dP}{dx} = 2x + 2(a-x) = +2a$. Proinde $\frac{dP}{dx}$ nunquam fit zero; & functio $P = 2a(x-a)$ nunquam fit $\begin{smallmatrix}\text{maxima}\\ \text{minima}\end{smallmatrix}$.

Exemplum quartum. Sit
$$P = x^3 - axx + bx + c$$
$$\frac{dP}{dx} = 3xx - 2ax + b. \text{ Facto } \frac{dP}{dx} = 0, \text{ fit } x = \frac{a \pm \sqrt{(aa-3b)}}{3}$$
$$\frac{ddP}{dx^2} = 6x - 2a \qquad \frac{ddP}{dx^2} = \pm \sqrt{(aa-3b)}.$$

Pro-

Proinde poſito $aa > 3b$, ſi $x = a + \sqrt{(aa-3b)}$, eſt P = minimum;
ſi $x = a - \sqrt{(aa-3b)}$, eſt P = maximum.

§. 172. Exponens differentialis $\frac{ddy}{dx^2}$ evaneſcat (ſive $\frac{dy}{dx}$ ſimul evaneſcat, ſive non). Hoc caſu fit

$$\frac{N'M'}{'N'M} = \mp \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} - \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \mp \frac{\Delta x^5}{1 \ldots 5} \cdot \frac{d^5y}{dx^5} - \ldots \text{ caſu concavitatis}$$

$$\frac{N'M'}{'N'M} = \pm \frac{\Delta x^3}{1.2.3} \cdot \frac{d^3y}{dx^3} + \frac{\Delta x^4}{1 \ldots 4} \cdot \frac{d^4y}{dx^4} \pm \frac{\Delta x^5}{1 \ldots 5} \cdot \frac{d^5y}{dx^5} + \ldots \text{ caſu convexitatis.}$$

Quare poſito, exponentem differentialem $\frac{d^3y}{dx^3}$ non evaneſcere, & ſeriem exponentium differentialium $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5}$ . . . ſigno $\frac{1}{(x-a)^n}$ non affici; non poteſt ſimul eſſe $\begin{matrix} N'P' > M'P' \\ 'N'P > 'M'P \end{matrix}$ vel $\begin{matrix} N'P' < M'P' \\ 'N'P < 'M'P \end{matrix}$. Proinde curva non poteſt eſſe ad utramque puncti M partem $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$; ſed punctum M concavum & convexum curvæ arcum invicem ſeparat, ſeu eſt punctum flexus contrarii curvæ.

Quodſi autem ſimul ſit $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$: ut curva ſit verſus axem $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$, debet eſſe $\frac{d^4y}{dx^4}$ $\begin{matrix}\text{negativus}\\ \text{poſitivus}\end{matrix}$; & proinde, ut functio P fiat omnium $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$, debet eſſe $\frac{d^4y}{dx^4}$ $\begin{matrix}\text{negativa}\\ \text{poſitiva}\end{matrix}$.

Exemplum primum. Sit $y = xx(a-x)$

$$\frac{dy}{dx} = x(2a-3x)$$

$$\frac{ddy}{dx^2} = 2a - 6x$$

$$\frac{d^3y}{dx^3} = -6$$

$\frac{dy}{dx}$ fit zero, ſi $\begin{matrix} x = 0 \\ 2a-3x = 0 \end{matrix}$, ſeu $\begin{matrix} x = 0 \\ x = \frac{2}{3}a \end{matrix}$. Tum vero

$\begin{matrix} \frac{ddy}{dx^2} = +2a \\ \frac{ddy}{dx^2} = -2a \end{matrix}$. Proinde abſciſſis $x = \begin{matrix} 0 \\ \frac{2}{3}a \end{matrix}$ reſpondent y $\begin{matrix}\text{minima}\\ \text{maxima}\end{matrix}$.

Sit

Sit autem $\frac{ddy}{dx^2} = 0$, & proinde $x = \frac{1}{3}a$; huic abſciſſæ reſpondet punctum flexus contrarii. Vide fig. 47. qua delineatur curſus curvæ, cujus æquatio eſt $y = xx(a - x)$.

Exemplum ſecundum. Sit

$$\begin{aligned} y &= x^3(a-x) \\ \frac{dy}{dx} &= xx(3a-4x) \\ \frac{ddy}{dx^2} &= 6x(a-2x) \\ \frac{d^3y}{dx^3} &= 6(a-4x) \\ \frac{d^4y}{dx^4} &= -24. \end{aligned}$$

Sit $\frac{dy}{dx} = 0$: erunt $x = \begin{matrix} 0 \\ 0 \\ \frac{3}{4}a \end{matrix}$; $\frac{ddy}{dx^2} = \begin{matrix} 0 \\ 0 \\ -\frac{9}{4}a \end{matrix}$, $\frac{d^3y}{dx^3} = \begin{matrix} +6a \\ +6a \\ -12a \end{matrix}$, $\frac{d^4y}{dx^4} = -24$.
Proinde abſciſſæ $x = 0$ reſpondet punctum flexus contrarii; & abſciſſæ $\frac{3}{4}a$ reſpondet maximum.

Sit autem $\frac{ddy}{dx^2} = 0$: erunt $x = \begin{matrix} 0 \\ \frac{1}{2}a \end{matrix}$, $\frac{d^3y}{dx^3} = \begin{matrix} +6a \\ -6a \end{matrix}$; quare iterum abſciſſis $\begin{matrix} 0 \\ \frac{1}{2}a \end{matrix}$ reſpondet punctum flexus contrarii.

In figura 48. delineatur curſus curvæ, cujus æquatio eſt $y = x^3(a-x)$.

§. 173. Simul ſint $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^4} = 0$, $\frac{d^4y}{dx^4} = 0$.

Erit $\begin{matrix} N'M' \\ 'N'M \end{matrix} = \mp \frac{\Delta x^5}{1\ldots5} \cdot \frac{d^5y}{dx^5} - \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} \mp \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} - \ldots$ caſu concavitatis

$\begin{matrix} N'M' \\ 'N'M \end{matrix} = \pm \frac{\Delta x^5}{1\ldots5} \cdot \frac{d^5y}{dx^5} + \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} \pm \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \ldots$ caſu convexitatis.

Proinde poſito: exponentem differentialem $\frac{d^5y}{dx^5}$ non evaneſcere, & ſeriem exponentium differentialium $\frac{d^5y}{dx^5}$, $\frac{d^6y}{dx^6}$, $\frac{d^7y}{dx^7} \ldots$ factorem impoſſibilem $\frac{1}{(x-a)^n}$ non comprehendere; nequit pro omnibus utut parvis mutationis Δx valoribus ſimul eſſe $\begin{matrix} N'P' \\ 'N'P \end{matrix} > \begin{matrix} M'P' \\ 'M'P \end{matrix}$ caſu concavitatis, & $\begin{matrix} N'P' \\ 'N'P \end{matrix} < \begin{matrix} M'P' \\ 'M'P \end{matrix}$ caſu concavitatis: quare hoc caſu punctum M eſt punctum flexus contrarii.

Sint autem ſimul $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = 0$, $\frac{d^5y}{dx^5} = 0$.

Quoniam eſt $\frac{y'}{y} = y \ldots + \frac{\Delta x^6}{1\ldots6} \cdot \frac{d^6y}{dx^6} \pm \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \pm \ldots$

ut functio y fiat omnium $\frac{\text{maxima}}{\text{minima}}$, debet eſſe $\frac{d^6y}{dx^6}$ $\frac{\text{negativus}}{\text{poſitivus}}$.

§. 174. Porro omnes exponentes differentiales $\frac{ddy}{dx^2} \ldots \frac{d^6y}{dx^6}$ ſimul evaneſcant;

unde $\frac{M'N'}{'M'N} = \ldots \mp \frac{'\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} - \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \mp \ldots$ caſu concavitatis,

$\frac{M'N'}{'M'N} = \ldots \pm \frac{\Delta x^7}{1\ldots7} \cdot \frac{d^7y}{dx^7} + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \pm \ldots$ caſu convexitatis.

Proinde $\frac{d^7y}{dx^7}$ non evaneſcente, non poteſt ſimul eſſe $\frac{M'N'}{'M'N} > \frac{M'P'}{'M'P}$ vel $\frac{M'N'}{'M'N} < \frac{M'P'}{'M'P}$; quare rurſus M eſt punctum flexus contrarii.

Sint autem $\frac{dy}{dx} = 0$, $\frac{ddy}{dx^2} = 0 \ldots \frac{d^7y}{dx^7} = 0$: quoniam eſt

$\frac{y'}{y} = y \ldots + \frac{\Delta x^8}{1\ldots8} \cdot \frac{d^8y}{dx^8} \pm \frac{\Delta x^9}{1\ldots9} \cdot \frac{d^9y}{dx^9} + \ldots$; ut y fiat omnium $\frac{\text{maxima}}{\text{minima}}$, debet eſſe $\frac{d^8y}{dx^8}$ $\frac{\text{negativus}}{\text{poſitivus}}$.

§. 175. Ex his ſatis ſuperque liquet methodus adhibenda, tam ut $\frac{\text{maxima}}{\text{minima}}$ functionis P (ſuppoſitionibus conſentaneæ), quam ut puncta flexus contrarii, ſi quæ occurrunt, determinentur. Et ſimul patet, eum eſſe utriusque hujus curvarum ſeu functionum ſymptomatis nexum mutuum, ut functiones ab uno ad alterum transeant; &, uno deficiente, alterum in locum ejus ſuccedat.

Generatim igitur regula hæc eſt. Si in ſerie exponentium differentialium $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3} \ldots \frac{d^my}{dx^m}$ impar horum exponentium numerus, a primo inde ordiundo, continue evaneſcit facto $x = a$; exponens autem par immediate ſequens non ſimul evaneſcit: functio propoſita omnium $\frac{\text{maxima}}{\text{minima}}$ eſt, prouti exponens hic par

par non evaneſcens eſt $\frac{\text{negativus}}{\text{poſitivus}}$. Quodſi vero, $\frac{dy}{dx}$ evaneſcente vel non evaneſcente poſito $x=a$, impar exponentium differentialium, qui illum ſequuntur, numerus ſimul evaneſcit, dum exponens differentialis ſequens non evaneſcit ſimul; abſciſſæ $x=a$ reſpondet punctum flexus contrarii.

Hinc poſito m numero integro poſitivo, ſi ſit $\frac{dP}{dx} = x^m Q$; abſciſſæ $x = 0$ reſpondet $\frac{\text{maximum}}{\text{minimum}}$ aut punctum flexus contrarii, prouti m impar eſt aut par.

1°. Sit $\frac{dP}{dx} = x^{2m-1} Q$

Erit $\frac{ddP}{dx^2} = x^{2m-2} Q'$

$$\frac{d^3 P}{dx^3} = x^{2m-3} Q''$$

$$\frac{d^4 P}{dx^4} = x^{2m-4} Q'''$$

$$\vdots$$

$$\frac{d^{2M-II} P}{dx^{2m-2}} = x^2 Q^{2M-III}$$

$$\frac{d^{2M-I} P}{dx^{2m-1}} = x Q^{2M-II}$$

$$\frac{d^{2M} P}{dx^{2m}} = Q^{2M-I}$$

Caſu igitur, quo m eſt impar, numerus impar exponentium differentialium ſucceſſivorum evaneſcit facto $x=0$, dum exponens differentialis ſequens non evaneſcit: proinde functio P eſt $\frac{\text{maxima}}{\text{minima}}$.

2°. Sit $\frac{dP}{dx} = x^{2m} Q$

Erit $\frac{ddP}{dx^2} = x^{2m-1} Q'$

$$\frac{d^3 P}{dx^3} = x^{2m-2} Q''$$

$$\frac{d^4 P}{dx^4} = x^{2m-2} Q'''$$

$$\vdots$$

$$\frac{d^{2M-II} P}{dx^{2m-2}} = x^3 Q^{2M-III}$$

$$\frac{d^{2M-I} P}{dx^{2m-1}} = x^2 Q^{2M-II}$$

$$\frac{d^{2M} P}{dx^{2m}} = x Q^{2M-I}$$

$$\frac{d^{2M+I} P}{dx^{2m+1}} = Q^{2M}.$$

Sed caſu, quo m eſt par, numerus par exponentium differentialium ſucceſſivorum evaneſcit facto $x=0$, dum exponens differentialis ſequens non evaneſcit: abſciſſæ igitur $x=0$ reſpondet punctum flexus contrarii.

§. 176. Ex æquationibus

$$\frac{dy}{dx} = 0 \quad \text{ſequitur} \quad y = C$$

$$\frac{ddP}{dx^2} = 0 \qquad y = Cx + C'$$

$$\frac{d^3y}{dx^3} = 0 \qquad y = \frac{1}{1.2}Cxx + C'x + C''$$

$$\frac{d^4y}{dx^4} = 0 \qquad y = \frac{1}{1.2.3}Cx^3 + \frac{1}{1.2}C'xx + C''x + C'''$$

$$\frac{d^5y}{dx^5} = 0 \qquad y = \frac{1}{1.2.3.4}Cx^4 + \frac{1}{1.2.3}C'x^3 + \frac{1}{1.2}C''xx + C'''x + C''''$$

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Proinde æquatio curvæ, $\genfrac{}{}{0pt}{}{\text{maximam}}{\text{minimam}}$ ordinatam habentis, ab æquatione lineæ rectæ axi parallelæ, vel ab æquationibus ordinis paris parabolarum, quibus y per potentias integras abſciſſæ x exprimitur, eo minus differt, quo propius puncta curvæ ad $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ punctum curvæ accedunt. Et æquatio curvæ, punctum flexus contrarii habentis, ab æquatione lineæ rectæ (axi parallelæ aut obliquæ) vel ab æquationibus ordinis imparis parabolarum, quibus y per poteſtates integras ipſius x exprimitur, eo minus differt, quo propius puncta curvæ ad flexus contrarii punctum accedunt.

§. 177. Qui ſymptomata inter curvarum, quibus flexus contrarii puncta aut $\genfrac{}{}{0pt}{}{\text{maximas}}{\text{minimas}}$ ordinatas admittunt, intercedit nexus mutuus, ſequenti etiam obſervatione illuſtratur.

Quæcunque de $\genfrac{}{}{0pt}{}{\text{maximo}}{\text{minimo}}$ functionis P dicuntur, vera ſunt de exponente differentiali, quando flexus contrarii punctum exiſtit. Proinde flexus contrarii determinatio reducitur ad determinationem $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ exponentis differentialis ſeu functionis $\frac{dy}{dx}$. Atqui (angulo coordinatarum poſito recto) $\frac{dy}{dx}$ tangens eſt trigonometrica anguli, quem recta curvam contingens facit cum axe. Itaque in flexus contrarii puncto tangens hæc trigonometrica omnium $\genfrac{}{}{0pt}{}{\text{maxima}}{\text{minima}}$ eſt; proinde etiam angulus ipſe fit omnium $\genfrac{}{}{0pt}{}{\text{maximus}}{\text{minimus}}$.

Pro-

Propoſitio hæc geometrice ſic ſtabilitur. Sit AMM' curva ab A inde usque ad M verſus axem concava, & ab M inde usque ad M' verſus axem convexa; ita ut M ſit punctum flexus contrarii. Ex M, M', $'M$ punctis agantur rectæ tangentes, quæ axi in T, T', $'T$ punctis occurrant; & tangens MT tangentibus $M'T'$, $'M'T$ in t' & $'t$ punctis occurrat. Caſu hoc in triangulis $TT't'$, $T'T't$ angulus T ſit internus, & minor angulis externis oppoſitis T', $'T$. Caſu autem, quo curva eſt ab A verſus M convexa, & ab M verſus M' concava, in iisdem triangulis angulus T ſit externus, & major angulis internis oppoſitis T', $'T$. Tandemque caſu, quo exponens differentialis $\frac{dy}{dx}$ evaneſcit, ſeu tangens curvæ in flexus contrarii puncto axi eſt parallela; anguli, quos tangentes, per puncta ex utraque flexus contrarii parte ſita ductæ, cum axe ad easdem ejus partes faciunt, ſimul ſunt acuti, & continue ad evaneſcentiam accedunt. Fig. 46.

Caſu, quo tam functio P, quam exponentes differentiales ſucceſſivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ . . . ſigno impoſſibilis ſeu infiniti non afficiuntur, expoſito; ad functiones progredior, quæ ſigno huic anſam præbent.

§. 178. Et primum quidem, ſi ſit $P = \varphi'x + \frac{\varphi x}{(x-a)^n}$; facto $x = a$, ſigni $\frac{\varphi x}{o^n}$ introductione docemur, functionem P fieri impoſſibilem. Hoc quippe caſu curva aſymptotum habet abſciſſæ $x = a$ reſpondentem, & ordinata puncto huic reſpondens eſt impoſſibilis. (Cap. IX.) Exponentes differentiales ſucceſſivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ . . . . impoſſibilitatis ſymbolis $\frac{1}{(x-a)^{n+1}}$, $\frac{1}{(x-a)^{n+2}}$, $\frac{1}{(x-a)^{n+3}}$ . . . . ſucceſſive etiam afficiuntur: quibus monemur, contradictorium eſſe de maximis ac minimis, de concavitate aut convexitate curvæ in puncto impoſſibili disquirere. Omnes curvæ aſymptoticæ, dum propius propiusque ad aſymptotum accedunt, ad eum tendunt ſtatum, ut tangentes earum fiant aſymptoto parallelæ; & curvatura ipſarum continue ad evaneſcentiam tendit, quam tamen nunquam aſſequitur.

§. 179. Miſſo hoc caſu, ad eos transeo, quibus functio P quidem impoſſibilis ſignum $\frac{1}{0}$ non involvit, ſed exponentes ejus differentiales illo afficiuntur.

Ut hoc eveniat, functio P factorem $(x-a)^n$ ſeu $(a-x)^n$ continere debet, cujus exponens n eſt numerus poſitivus non integer; & magnitudo hujus exponentis determinat ordinem exponentis differentialis, qui primus impoſſibilis ſignum involvet.

Exempla. Sit n fractio vera, ſeu $n \begin{smallmatrix} > 0 \\ < 1 \end{smallmatrix}$: exponens $\frac{dP}{dx}$ caſu $x = a$ afficietur ſigno impoſſibilis $\frac{1}{0^{1-n}}$: quo docemur: tangentem per punctum curvæ, quod abſciſſæ $x = a$ reſpondet, ductam fieri rectis axi ordinatim applicatis parallelam, ſeu tangentem hanc axi eſſe perpendicularem.

Sit n fractio ſpuria, & quidem $n \begin{smallmatrix} > 1 \\ < 2 \end{smallmatrix}$: tum exponens differentialis $\frac{ddP}{dx^2}$ impoſſibilis ſigno $\frac{1}{0^{2-n}}$ primus afficitur.

Sit $n \begin{smallmatrix} > 2 \\ < 3 \end{smallmatrix}$: exponens differentialis $\frac{d^3P}{dx^3}$ ille eſt, qui impoſſibilis ſigno $\frac{1}{0^{3-n}}$ primus afficitur.

Univerſim ſit $n \begin{smallmatrix} > m-1 \\ < m \end{smallmatrix}$: erit exponens differentialis $\frac{d^mP}{dx^m}$ is, qui impoſſibilis ſigno $\frac{1}{0^{m-n}}$ primus afficitur.

§. 180. Ut, quæ ad caſum hunc pertinent, eo diſtinctius tradam: a caſibus ordiar ſimpliciſſimis, iis nempe, quibus $y = x^n$, ſeu a curvis parabolicis æquatione hac deſignatis.

Et primo quidem ſit n numerus fractus verus $\frac{p}{q}$, ad ſimpliciſſimos redactus terminos, ſeu cujus termini p & q diviſorem communem non habent (quod poſtea ſemper ſupponetur).

Tres occurrunt diſtinguendi caſus. Poteſt quippe fractionis $\frac{p}{q}$ terminus unus eſſe par, quo caſu alter erit impar; & tum par erit vel numerator p, vel denominator q: vel uterque terminus p, q poteſt eſſe impar.

Primus caſus. Sit p numerus par, q impar.

Functionis $x^{\frac{p}{q}}$ idem eſt valor, ſeu x fiat negativa, ſeu poſitiva: proinde duabus abſciſſis æqualibus, quarum una poſitiva, altera negativa eſt, duæ reſpondent

dent ordinatæ æquales poſitivæ; ideoque curva duobus conſtat ramis invicem congruentibus ad easdem axis partes, ſed ad diverſas verticis partes ſitis.

Quoniam $y = x^n$:

$$\text{ſunt } \frac{dy}{dx} = + nx^{n-1} = + n \cdot \frac{1}{x^{1-n}}$$

$$\frac{ddy}{dx^2} = - n \cdot 1 - n \cdot \frac{1}{x^{2-n}}$$

$$\frac{d^3y}{dx^3} = + n \cdot 1 - n \cdot 2 - n \cdot \frac{1}{x^{3-n}}$$

$$\frac{d^4y}{dx^4} = - n \cdot 1 - n \ldots 3 - n \cdot \frac{1}{x^{4-n}}$$

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Proinde, quamdiu x non eſt zero, curva verſus axem concava eſt, propter $\frac{ddy}{dx^2} = - n \cdot 1 - n \frac{1}{x^{2-n}}$: & quoniam $y = x^n$; quo major eſt x, ſive poſitiva, ſive negativa, eo major eſt y: ideoque facta $x = 0$, $y = 0$ eſt omnium minima.

Factis $x = 0$, $y = 0$, eſt $\frac{dy}{dx} = \frac{1}{0}$; proinde communis duorum ramorum tangens in vertice eſt axi perpendicularis, ſeu rectis axi ordinatim applicatis parallela.

Duo igitur rami cuſpidem ad verticem formant, plus minusve acutam tum pro vario exponente n, tum & pro varia parametro, qua fit $y = p^{1-n} x^n$.

Figura 49. 1°. ſiſtit curvam, cujus æquatio eſt $y = x^{\frac{2}{5}}$; & fig. 49. 2°. curvam, cujus æquatio eſt $y = x^{\frac{4}{5}}$.

Caſus ſecundus. Sit p numerus impar, q par.

Hoc caſu x non poteſt eſſe negativa. Et quoniam $y = x^{\frac{p}{q}}$; ob q numerum parem, eidem abſciſſæ duæ reſpondent ordinatæ æquales, una poſitiva, altera negativa: unde curva duobus conſtat ramis invicem congruentibus, ad diverſas axis partes, ſed ad easdem verticis partes ſitis. Caſu hoc ordinata zero, abſciſſæ zero reſpondens, minor eſt rectis axi ordinatim applicatis in regione ordinatarum poſitivarum ſitis; major autem in calculo algebraico cenſetur applicatis in regione ordinatarum negativarum ſitis: itaque ordinata hæc neque maxima

xima neque minima judicatur. Sed abſciſſa zero minor eſt abſciſſis poſitivis; ideoque hoc caſu datur minimum abſciſſarum, ſed nullum $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$ ordinatarum. Porro recta curvam in vertice contingens parallela eſt rectis axi ordinatim applicatis.

Quoniam $y = \pm x^n$: ſunt $\frac{dy}{dx} = \pm n \frac{1}{x^{1-n}}$

$\frac{ddy}{dx^2} = \mp n . 1 - n \frac{1}{x^{2-n}}$. Proinde ramus ad partes ordinatarum poſitivarum ſitus verſus axem concavus eſt; ſed ramus ad partes ordinatarum negativarum jacens verſus axem convexus eſt, quatenus ad easdem cum priore axis partes refertur; ſeu hi duo rami ſunt $\begin{matrix}\text{unus concavus}\\\text{alter convexus}\end{matrix}$ verſus easdem partes cujusvis rectæ axi parallelæ. Vertex ideo ſpectari poſſet tanquam punctum flexus contrarii, quatenus curva ad rectam axi parallelam refertur. Sed cum duo rami ſimiliter flectantur verſus rectam axi perpendicularem; v. gr. verſus rectam, quæ curvam in vertice tangit: uſu receptum hoc caſu non eſt, verticem flexus contrarii punctum vocare; ſed potius vertex ut $\begin{matrix}\text{maximi}\\\text{minimi}\end{matrix}$ punctum, quod ad abſciſſas, ſpectatur.

Fig. 50. $\begin{matrix}1^\circ.\\2^\circ.\end{matrix}$ curvam exhibet, cujus æquatio eſt $y \begin{matrix}= x^{\frac{3}{10}}\\= x^{\frac{7}{10}}\end{matrix}$.

Ad hanc claſſem pertinet parabola conica, cujus æquatio eſt $y = x^{\frac{1}{2}}$.

Caſus tertius. Numeri p, q ambo ſint impares.

Hoc caſu abſciſſis æqualibus $\begin{matrix}\text{poſitivis}\\\text{negativis}\end{matrix}$ reſpondent ordinatæ æquales etiam $\begin{matrix}\text{poſitivæ}\\\text{negativæ}\end{matrix}$. Curva igitur duobus conſtat ramis invicem æqualibus, ad diverſas tam axis quam verticis partes ſitis; ideoque ordinata vertici reſpondens nec maxima nec minima eſt cenſenda.

Quoniam $y = x^n$, $\frac{dy}{dx} = n \frac{1}{x^{1-n}}$; proinde recta curvam in vertice contingens parallela eſt rectis axi ordinatim applicatis.

Porro $\frac{ddy}{dx^2} = - n . 1 - n \frac{1}{x^{2-n}}$; & quoniam numeri p & q ambo ſunt impares, etiam fractionis $2 - n$ ambo termini impares ſunt. Proinde ſumta x poſitiva, $\frac{ddy}{dx^2}$ eſt negativa; ideoque curva verſus axem concava eſt: ſumta autem x negativa,

gativa, $\frac{1}{x^{2-n}}$ pariter negativa eſt, proinde $\frac{ddy}{dx^2}$ fit poſitiva: quare in regione ordinatarum negativarum curva eſt convexa, quatenus easdem cum ramo priore partes axis reſpicit, ſeu quatenus refertur ad rectam aliquam axi parallelam. Proinde hoc caſu vertex curvæ eſt punctum flexus contrarii.

Figuris 51. N. $\frac{1,}{2.}$ delineantur ductus curvarum, quarum æquationes ſunt

$$y = \begin{cases} x^{\frac{3}{5}}. \\ x^{\frac{1}{5}} \end{cases}$$

Jam ſit n fractio ſpuria ſeu major unitate. Caſus hic ad præcedentem ſemper reducitur. Cum enim ſit n fractio ſpuria, $\frac{1}{n}$ eſt fractio vera: atqui quoniam $y = x^n$, eſt $y^{\frac{1}{n}} = x$: proinde, permutatis ordinatis, ſeu curva ad tangentem per verticem ductam tanquam axem relata, caſus hic ad priorem reducitur.

Exempla. Sit $y = x^{\frac{3}{2}}$: erit $x = y^{\frac{2}{3}}$

$y = x^{\frac{9}{5}}$: $x = y^{\frac{5}{9}}$

$y = x^{\frac{6}{5}}$: $x = y^{\frac{5}{6}}$.

§. 181. Enodato caſu hoc ſimpliciſſimo, facilior erit diſcuſſio ceterorum caſuum, quibus æquatio propoſita magis compoſita eſt.

Sit nempe $y = \varphi x \cdot x^n$, n denotante numerum fractum, & φx functionem variabilis x, quæ facta $x = 0$ non evaneſcit.

Erunt

$$\frac{dy}{dx} = \frac{d\varphi x}{dx} \cdot x^n + n\varphi x \cdot x^{n-1}$$

$$\frac{ddy}{dx^2} = \frac{d^2\varphi x}{dx^2} \cdot x^n + 2n \cdot \frac{d\varphi x}{dx} \cdot x^{n-1} + n \cdot n-1\,\varphi x \cdot x^{n-2}$$

$$\frac{d^3y}{dx^3} = \frac{d^3\varphi x}{dx^3} \cdot x^n + 3n \cdot \frac{d^2\varphi x}{dx^2} \cdot x^{n-1} + 3n \cdot n-1 \cdot \frac{d\varphi x}{dx} \cdot x^{n-2} + n \ldots n-2\,\varphi x \cdot x^{n-3}$$

d^4y

$$\frac{d^4y}{dx^4} = \frac{d^4\varphi x}{dx^4} . x^{n}$$
$$+ 4n . \frac{d^3\varphi x}{dx^3} . x^{n-1}$$
$$+ 6n . n-1 . \frac{d^2\varphi x}{dx^2} . x^{n-2}$$
$$+ 4n \ldots n-2 . \frac{d\varphi x}{dx} . x^{n-3}$$
$$+ n \ldots n-3\, \varphi x . x^{n-4}$$

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I°. Sit n numerus fractus verus: erit $\frac{dy}{dx} = \frac{d\varphi x}{dx} . x^{n}$ $+ n\varphi x \frac{1}{x^{1-n}}$. Proinde hoc caſu, facta $x = 0$, fit $\frac{dy}{dx} = n\varphi x . \frac{1}{0^{1-n}}$; ſeu tangens ad verticem parallela eſt rectis axi ordinatim applicatis.

Tum quo minor eſt x, eo minus æquatio differentialis curvæ differt ab æquatione $\frac{dy}{dx} = n\varphi x . \frac{1}{x^{1-n}}$: & quoniam φx non involvit factorem x, functio hæc formam habet $A + Bx + Cx^2 + Dx^3 + \ldots$; & imminuta x limes ejus eſt A. Proinde æquatio differentialis $\frac{dy}{dx} = n\varphi x . \frac{1}{x^{1-n}}$ propius ſemper propiusque accedit ad æquationem differentialem parabolicam $\frac{dy}{dx} = nA \frac{1}{x^{1-n}}$; & curva ipſa ad parabolam, cujus æquatio eſt $y = Ax^{n}$: ideoque, facta $x = 0$, y minima eſt, ſi n ſit numerus fractus, cujus numerator eſt par; y nec maxima nec minima eſt, ſi n ſit numerus fractus, cujus denominator eſt par. Denique abſciſſæ $x = 0$ reſpondet punctum flexus contrarii, ſi n ſit numerus fractus, cujus tam numerator quam denominator ſunt numeri impares.

II°. Sit n numerus fractus ſpurius $\begin{matrix} > 1 \\ < 2 \end{matrix}$.

Hinc $\frac{dy}{dx} = \frac{d\varphi x}{dx} x^{n}$
$+ n\varphi x . x^{n-1}$.

Facta

Facta $x=0$, eſt $\frac{dy}{dx}=0$; proinde recta curvam in vertice contingens parallela eſt axi. Porro $\frac{ddy}{dx^2}=\frac{d^2\varphi x}{dx^2}x^n$

$$+\ 2n\frac{d\varphi x}{dx}x^{n-1}$$

$$+\ n.n-1\,\varphi x\frac{1}{x^{2-n}}.$$

Igitur, imminuta x, exponens differentialis ſecundi ordinis $\frac{ddy}{dx^2}$ propius ſemper propiusque accedit ad hunc valorem, ut ſit $\frac{ddy}{dx^2}=n.n-1\,\varphi x\frac{1}{x^{2-n}}=n.n-1\,(A+Bx+Cx^2+Dx^3+\ldots).\frac{1}{x^{2-n}}$; & proinde æquatio curvæ propius ſemper propiusque accedit ad æquationem parabolæ, cujus æquatio differentialis ſecundi ordinis eſt $n.n-1.\frac{1}{x^{2-n}}$.

1°. Sit $n=1+\frac{2p+1}{2q}=\frac{2p+2q+1}{2q}$

$\frac{1}{n}=\frac{2q}{2p+2q+1}$; proinde forma hæc reducitur ad primum caſum (§. 180.), atque x omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$ fit.

2°. Sit $n=1+\frac{2p}{2q+1}=\frac{2p+2q+1}{2q+1}$, $\frac{1}{n}=\frac{2q+1}{2p+2q+1}$; igitur (§. 180. caſ. 3.) abſciſſæ $x=0$ reſpondet punctum flexus contrarii.

3°. Sit $n=1+\frac{2p+1}{2q+1}=\frac{2p+2q+2}{2q+1}$, $\frac{1}{n}=\frac{2q+1}{2p+2q+2}$; igitur (§. 180. caſ. 2.) abſciſſæ $x=0$ reſpondet $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$ ordinatæ y.

III°. Sit n numerus fractus ſpurius $\begin{matrix}>2\\<3\end{matrix}$.

Hinc $\frac{dy}{dx}=\frac{d\varphi x}{dx}.x^n$

$$+\ n\varphi x.x^{n-1}$$

$$\frac{ddy}{dx^2}=\frac{d^2\varphi x}{dx^2}.x^n$$

$$+\ 2n\frac{d\varphi x}{dx}.x^{n-1}$$

$$+\ n.n-1\,\varphi x.x^{n-2}$$

 d^3y

$$\frac{d^3y}{dx^3} = \frac{d^3\varphi x}{dx^3} . x^n$$
$$+ 3n\frac{d^2\varphi x}{dx^2} . x^{n-1}$$
$$+ 3n.n-1\frac{d\varphi x}{dx} . x^{n-2}$$
$$+ n\ldots n-2\varphi x . \frac{1}{x^{3-n}}.$$

Proinde, imminuta x, exponens differentialis $\frac{d^3y}{dx^3}$ propius ſemper propiusque ad valorem $n.n-1.n-2A\frac{1}{x^{3-n}}$ accedit: ideoque caſus hic reducitur ad contemplationem curvæ parabolicæ, cujus æquatio eſt $y = x^n$.

1°. Sit $n = 2 + \frac{2p}{2q+1} = \frac{4q+2p+2}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4q+2p+2}$: unde (§. 180. caſ. 2.) facta $x = 0$, y eſt omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$.

2°. Sit $n = 2 + \frac{2p+1}{2q} = \frac{4q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4q+2p+1}$: unde (§. 180. caſ. 1.) facta $x = 0$, x eſt omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$.

3°. Sit $n = 2 + \frac{2p+1}{2q+1} = \frac{4q+2p+3}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4q+2p+3}$: unde (§. 180. caſ. 3.) abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

IV. Sit n numerus fractus $\begin{matrix}>3\\<4\end{matrix}$

$$\frac{d^4y}{dx^4} = \frac{d^4\varphi x}{dx^4} . x^n$$
$$+ 4n\frac{d^3\varphi x}{dx^3} . x^{n-1}$$
$$+ 6n.n-1\frac{d^2\varphi x}{dx^2} . x^{n-2}$$
$$+ 4n\ldots n-2\frac{d\varphi x}{dx} . x^{n-4}$$
$$+ n\ldots n-3\varphi x\frac{1}{x^{4-n}}.$$

Quare, imminuta x, exponens differentialis $\frac{d^4y}{dx^4}$ propius ſemper propiusque accedit

cedit

cedit ad exponentem differentialem quarti gradus parabolæ, nempe $\frac{d^4y}{dx^4} = n \dots n-3\, A \frac{1}{x^{n-4}}$.

1°. Sit $n = 3 + \frac{2p}{2q+1} = \frac{6q+2p+3}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{6q+2p+3}$; proinde abſciſſæ $x=0$ reſpondet punctum flexus contrarii (§. 180. caſ. 3.).

2°. Sit $n = 3 + \frac{2p+1}{2q} = \frac{6q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{6q+2p+1}$; proinde abſciſſa $x=0$ eſt omnium $\frac{\text{maxima}}{\text{minima}}$ (§. 180. caſ. 1.).

3°. Sit $n = 3 + \frac{2p+1}{2q+1} = \frac{6q+2p+4}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{6q+2p+4}$; proinde abſciſſæ $x=0$ reſpondet ordinata omnium $\frac{\text{maxima}}{\text{minima}}$.

V. Sit $n \begin{smallmatrix} > 4 \\ < 5 \end{smallmatrix}$: erit

$$\begin{aligned} \frac{d^5y}{dx^5} = & \frac{d^5\varphi x}{dx^5} . x^n \\ & + 5n \frac{d^4\varphi x}{dx^4} . x^{n-1} \\ & + 10n.n-1 \frac{d^3\varphi x}{dx^3} . x^{n-2} \\ & + 10n.n-1.n-2 \frac{d^2\varphi x}{dx^2} . x^{n-3} \\ & + 5n \dots n-3 \frac{d\varphi x}{dx} . x^{n-4} \\ & + n \dots n-4\, \varphi x . \frac{1}{x^{5-n}}. \end{aligned}$$

Proinde, imminuta x, æquatio curvæ per exponentem differentialem $\frac{d^5y}{dx^5}$ determinata propius ſemper propiusque accedit ad $\frac{d^5y}{dx^5} = n \dots n-4\, A \frac{1}{x^{5-n}}$.

1°. Sit $n = 4 + \frac{2p}{2q+1} = \frac{8q+2p+4}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{8q+2p+4}$; abſciſſæ $x=0$ reſpondet ordinata y omnium $\frac{\text{maxima}}{\text{minima}}$.

2°. Sit $n = 4 + \frac{2p+1}{2q} = \frac{8q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{8q+2p+1}$; $x=0$ eſt omnium $\frac{\text{maxima}}{\text{minima}}$.

3°. Sit

3°. Sit $n = 4 + \frac{2p+1}{2q+1} = \frac{8q+2p+5}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{8q+2p+5}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

Ab exemplis his facilis eſt tranſitus ad regulam generalem, qua curvarum æquatione $y = \varphi x . x^n$ definitarum ſymptomata caſu $x = 0$ determinantur.

1°. Si $n = 2m + \frac{2p}{2q+1} = \frac{4qm+2m+2p}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2p}$; y minima eſt caſu $x = 0$.

2°. Si $n = 2m + \frac{2p+1}{2q} = \frac{4qm+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4qm+2p+1}$; $x = 0$ minima eſt.

3°. Si $n = 2m + \frac{2p+1}{2q+1} = \frac{4qm+2m+2p+1}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2p+1}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

4°. Si $n = 2m+1 + \frac{2p}{2q+1} = \frac{4qm+2m+2q+2p+1}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2q+2p+1}$; abſciſſæ $x = 0$ reſpondet punctum flexus contrarii.

5°. Si $n = 2m+1 + \frac{2p+1}{2q} = \frac{4qm+2q+2p+1}{2q}$, $\frac{1}{n} = \frac{2q}{4qm+2q+2p+1}$; abſciſſa $x = 0$ minima eſt.

6°. Si $n = 2m+1 + \frac{2p+1}{2q+1} = \frac{4qm+2m+2q+2p+2}{2q+1}$, $\frac{1}{n} = \frac{2q+1}{4qm+2m+2q+2p+2}$; y minima eſt caſu $x = 0$.

Eadem ratiocinia applicantur functionibus $y = \varphi' x \pm \varphi x . x^n$, in quibus n eſt numerus fractus poſitivus; & $\varphi' x$, φx functiones ſunt variabilis x non evaneſcentes caſu $x = 0$. Quoniam autem ſingulare hoc functionum genus in matheſi inprimis applicata quam rariſſime occurrit; ſufficiat generale, ad quod exigi debent, principium expoſuiſſe.

Huc usque tradita variis exemplis illuſtrabo.

§. 182. *Exemplum* 1. Sit $P = \sqrt{(aa+xx)} - \frac{p}{q} x$

$$\frac{dP}{dx} = \frac{x}{\sqrt{aa+xx}} - \frac{p}{q}$$

$$\frac{ddP}{dx^2} = \frac{1}{\sqrt{aa+xx}} - \frac{xx}{(aa+xx)^{\frac{3}{2}}} = + \frac{aa}{(aa+xx)^{\frac{3}{2}}}.$$

Facto

Facto $\frac{dP}{dx} = 0$, eſt $xx = \frac{pp}{qq}(aa+xx)$, $x = a \times \frac{p}{\sqrt{(qq-pp)}}$.

Ut problema poſſibile ſit; oportet, ſit $q > p$: & facto $x = a . \frac{p}{\sqrt{(qq-pp)}}$ P eſt omnium minima, propter $\frac{ddP}{dx^2}$ ſemper poſitivum.

Exemplum 2. Sit $P = x^x$

$$\log. P = x \log. x$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = \log. x + 1$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = \frac{1}{x} \text{ (propter } \frac{dP}{dx} = 0\text{)}.$$

Facto $\log. x + 1 = 0$, eſt $\log. x = -1$, ſeu $x = e^{-1} = \frac{1}{e}$; $P = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \sqrt[e]{\frac{1}{e}}$.

Exemplum 3. Sit $P = \text{ſin.}^m x \, \text{ſin.}^n (\varphi - x)$

$$\log. P = m \log.\text{ſin.} x + n \log.\text{ſin.} (\varphi - x)$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \cot. x - n \cot. (\varphi - x)$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -\tfrac{1}{2} m \,\text{coſec.} 2x - \tfrac{1}{2} n \,\text{coſec.} (2\varphi - 2x).$$

Proinde P maximum eſt, quando $m \cot. x = n \cot. (\varphi - x)$; unde $\text{ſin.}(2x - \varphi) = \frac{n-m}{n+m} \text{ſin.} \varphi$.

Exemplum 4. Sit $P = \text{tang.}^m x \, \text{tang.}^n (\varphi - x)$

$$\log. P = m \log.\text{tang.} x + n \log.\text{tang.} (\varphi - x)$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \,\text{coſec.} x - n \,\text{coſec.} (\varphi - x)$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -m \cot. x \,\text{coſec.} x - n \,\text{coſec.} (\varphi - x) \cot. (\varphi - x).$$

Proinde facto $m \,\text{coſec.} x = n \,\text{coſec.} (\varphi - x)$, unde $\text{tang.}(\tfrac{1}{2}\varphi - x) = \frac{n-m}{n+m} \text{tang.} \tfrac{1}{2}\varphi$; functio P eſt omnium maxima.

Exemplum 4. Sit $P = x^m \text{ſin.}^n x$

$$\log. P = m \log. x + n \log.\text{ſin.} x$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m . \frac{1}{x} + n \cot. x$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -m . \frac{1}{xx} - n \,\text{coſec.}^2 x.$$

Pro-

Proinde ſacto $\frac{m}{n} = -x \cot. x = x \cot. 180^\circ - x$, ſeu $x = \frac{m}{n} \tang. 180^\circ - x$; P eſt omnium maxima.

Exemplum 6. Sit $P = \text{ſin.}\, x\, \text{ſin.}\, mx$

$$\log. P = \log.\text{ſin.}\, x + \log.\, \text{ſin.}\, mx$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = \cot.\, x + m \cot.\, mx$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -\,\text{coſec.}^2 x - mm\, \text{coſec.}^2 mx.$$

Proinde facto $\cot.\, 180^\circ - x = m \cot.\, mx$, functio P eſt omnium maxima.

Exemplum 7. Sit $P = \frac{(a+x)^m}{(b+x)^n}$. Facta $x = -b$, nullus eſt functionis P limes quod ad magnitudinem.

$$\log. P = m \log. (a+x) - n \log. (b+x)$$

$$\frac{dP}{dx} \cdot \frac{1}{P} = m \frac{1}{a+x} - n \cdot \frac{1}{b+x}$$

$$\frac{ddP}{dx^2} \cdot \frac{1}{P} = -\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2}.$$

1°. Sit $n = m$: facto $a + x = b + x$, omnes exponentes differentiales ſucceſſivi functionis P evaneſcunt; prouti nullus eſt functionis hujus limes, tum quod ad magnitudinem, tum quod ad parvitatem.

2°. Sit $m > n$: facto $\frac{m}{a+x} = \frac{n}{b+x}$, eſt $\frac{m}{(a+x)^2} = \frac{nn}{m} \cdot \frac{1}{(b+x)^2}$, et $-\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2} = \frac{n}{(b+x)^2}\left(1 - \frac{n}{m}\right) = + \frac{1}{(b+x)^2} \times \frac{n(m-n)}{m}$; proinde functio P eſt omnium minima.

3°. Sit $m < n$: tum $-\frac{m}{(a+x)^2} + \frac{n}{(b+x)^2} = -\frac{n}{(b+x)^2} \times \frac{n-m}{m}$; proinde functio P eſt omnium maxima.

§. 183. Hactenus ſuppoſui, functionem P quantitatis mutabilis x ita exprimi per hanc quantitatem, ut functio hæc ſola ſit in uno æquationis membro, neque alterum ingrediatur. Fieri autem etiam poteſt, ut cognitio relationis functionis hujus ad quantitatem mutabilem x ab æquationis alicujus ſolutione pendeat; quo caſu functio P ſit multiformis.

Caſus

Caſus hujus difficultas unice refertur ad imperfectum æquationum doctrinæ ſtatum. Etenim ſi ſolutio æquationum in promtu eſſet: reducta æquatione, a qua functionis P & quantitatis x relatio mutua pendet; radices ejus totidem exhiberent functionis P per variabilem x expreſſiones, & $\begin{matrix}\text{maximorum}\\ \text{minimorum}\end{matrix}$ ejus valorum determinatio reduceretur ad totidem functiones uniformes, quot æquatio propoſita habet radices. Hinc functiones biformes, quæ ab æquatione ſecundi gradus pendent, facile reducuntur ad duas functiones uniformes.

Argumentum hoc exemplis quibusdam aptiſſime declarabitur.

Exemplum 1. Sit y functio triformis ipſius x, quæ exprimatur æquatione $y^3-pxy+x^3=0$; & quæratur valor omnium $\begin{matrix}\text{maximus}\\ \text{minimus}\end{matrix}$ functionis y.

Quoniam $y^3-pxy+x^3=0$;

$3yy\frac{dy}{dx}-px\frac{dy}{dx}-py+3xx=0$: facto $\frac{dy}{dx}=0$, erit $py=3xx$, & $y=\frac{3xx}{p}$. Valor hic in æquatione propoſita ſubſtituatur: fiet $\frac{27x^6}{p^3}-2x^3=0$, $27x^6=2p^3x^3$;

hinc $3xx=px\sqrt[3]{2}$, $x=\begin{matrix}0\\ \frac{1}{3}p\sqrt[3]{2}\end{matrix}$, $y=\begin{matrix}0\\ \frac{1}{3}p\sqrt[3]{4}\end{matrix}$.

Quoniam $3yy\frac{dy}{dx}-px\frac{dy}{dx}=py-3xx$: facto $\frac{dy}{dx}=0$,

eſt $3yy\frac{ddy}{dx^2}-px\frac{ddy}{dx^2}=-6x$; æquatio identica, factis $\begin{matrix}x\\ y\end{matrix}=0$.

Porro $\begin{matrix}3yy\frac{d^3y}{dx^3}\\ -px\frac{d^3y}{dx^3}\end{matrix}-p\frac{ddy}{dx^2}=-6$: unde poſitis $\begin{matrix}x\\ y\end{matrix}=0$, fit $-p\frac{ddy}{dx^2}=-6$,

$\frac{ddy}{dx^2}=+\frac{6}{p}$; quare $y=0$ eſt omnium minima.

Sit autem $\begin{matrix}x=\frac{1}{3}p\sqrt[3]{2}\\ y=\frac{1}{3}p\sqrt[3]{4}\end{matrix}$; tum $\frac{ddy}{dx^2}(\frac{2}{3}pp\sqrt[3]{2}-\frac{1}{3}pp\sqrt[3]{2})=-2p\sqrt[3]{2}$

ſeu $\frac{ddy}{dx^2}=-\frac{3}{p}$; proinde caſu hoc y eſt omnium maxima.

Exemplum 2. Sit $y^4-4a^2xy+x^4=0$;

itaque $4y^3\frac{dy}{dx}-4a^2x\frac{dy}{dx}-4a^2y+4x^3=0$: facto $\frac{dy}{dx}=0$, eſt $y=\frac{x^3}{a^3}$.

Hinc

Hinc $\frac{x^{12}}{a^8} - 3x^4 = 0$, $x^4(\frac{x^8}{a^8} - 3) = 0$, $x^4(x^8 - 3a^8) = 0$, $x^4(x^4 + a^4\sqrt{3})(x^4 - a^4\sqrt{3}) = 0$, $x^4(x^4 + a^4\sqrt{3})(xx + aa\sqrt[4]{3})(xx - aa\sqrt[4]{3}) = 0$; cujus æquationes radices reales ſunt $x = \begin{cases} 0 \\ +a\sqrt[8]{3} \\ -a\sqrt[8]{3} \end{cases}$: unde $y = \begin{cases} 0 \\ a\sqrt[8]{27} \\ -a\sqrt[8]{27} \end{cases}$.

Quoniam $y^3\frac{dy}{dx} - a^2x\frac{dy}{dx} = a^2y - x^3$: poſito $\frac{dy}{dx} = 0$, fit

$$y^3\frac{ddy}{dx^2} - a^2x\frac{ddy}{dx^2} = -3xx.$$

Fiat $y = +a\sqrt[8]{27}$: erit $2a^3\sqrt[8]{3} \times \frac{ddy}{dx^2} = -3a^3\sqrt[4]{3}$; proinde y eſt maxima.

Fiat $y = -a\sqrt[8]{27}$: erit $-4a^3\sqrt[8]{3} \times \frac{ddy}{dx^2} = -3a^3\sqrt[4]{3}$; igitur y eſt minima.

Ex his exemplis ſequitur regula generalis. Differentietur æquatio propoſita; facto $\frac{dy}{dx} = 0$, inde eliciatur (quoad fieri poteſt) valor ipſius y per x expreſſæ, qui in æquatione propoſita ſubſtituatur. Æquatione hac reducta, notentur valores ipſius x inde eruti, & qui illis reſpondent valores ipſius y, qui exhibebunt $\frac{\text{maximum}}{\text{minimum}}$, ſi quod locum habet. Quod ut definiatur: ex æquatione differentiali eliciatur exponens differentialis $\frac{ddy}{dx^2}$ (omiſſis in altera differentiatione terminis exponente differentiali $\frac{dy}{dx} = 0$ affectis); ex quo, ſubſtitutis ipſarum x & y valoribus inventis, dijudicabitur (ſi non evaneſcat): utrum valoribus inventis $\frac{\text{maximum}}{\text{minimum}}$ reſpondeat? Quodſi autem exponens differentialis $\frac{ddy}{dx^2}$ evaneſcit; recurrendum eſt ad exponentes differentiales ulteriorum ordinum.

Hinc patet: quantopere methodus hæc ab æquationum ſolutione pendeat; & proinde quot difficultatibus poſſit caſibus, præſertim complexis, eſſe obnoxia.

§. 184. Methodus huc usque tradita functionum alicujus quantitatis mutabilis $\frac{\text{maxima}}{\text{minima}}$ determinandi omnium eſt univerſaliſſima. Occurrunt tamen caſus, quibus alium tenere modum minus algebraicum præſtat, & qui in matheſi præcipue applicata felici ſucceſſu frequenter uſurpatur.

Metho-

Methodus hæc ſequenti nititur principio. Si functio quantitatis alicujus mutabilis $\frac{\text{maxima}}{\text{minima}}$ fit pro dato quodam variabilis x valore a; duo ejusdem functionis valores invicem æquales aſſignari poſſunt, quorum unus majori, alter minori quantitatis mutabilis x valori reſpondet.

Principium hoc apprime declarat theoria curvarum atque ordinatarum $\frac{\text{maximarum}}{\text{minimarum}}$.

Primo tangens curvæ, per extremum ordinatæ omnium $\frac{\text{maximæ}}{\text{minimæ}}$ ducta, ſit axi parallela; tangens hæc a puncto contactus inde, motu ſibi parallelo, aut axi propius admoveatur, aut ab ipſo recedat, prouti curva eſt verſus axem $\frac{\text{concava}}{\text{convexa}}$: ita tangens hæc curvæ ex utraque ordinatæ omnium $\frac{\text{maximæ}}{\text{minimæ}}$ parte occurret; ordinatisque æqualibus a punctis occurſus ductis æquales reſpondent functionis, per ordinatas curvæ deſignatæ, valores. Sit dein tangens curvæ, per extremum ordinatæ $\frac{\text{maximæ}}{\text{minimæ}}$ ducta, ordinatis parallela: quoniam curva duobus conſtat ramis in hoc puncto ſe invicem contingentibus, & ad easdem axis partes ſitis; recta per hoc punctum axi parallela, & motu ſibi parallelo progreſſa a puncto illo inde verſus partes, ad quas curva jacet, utrique etiam ramo occurret; & punctis occurſus duæ reſpondebunt ordinatæ invicem æquales, ſeu duo functionis ordinatis curvæ proportionalis valores invicem æquales.

Applicationem principii hujus exemplis aliquot illuſtrabo, a ſimpliciſſimis ordiendo.

Exemplum I. Sit AB recta in puncto Z in duas partes ſic dividenda, ut rectangulum $AZ \times ZB$ ſit omnium maximum. Fig. 52.

Sint X & X' duo puncta ad utramque puncti Z partem ſita, quibus reſpondeant rectangula $AX \times XB$, $AX' \times X'B$ invicem æqualia.

Quoniam $AX \times XB = AX' \times X'B$: eſt $AX : AX' = X'B : XB$

$$\text{hinc } AX : XX' = X'B : XX'$$
$$\text{et } AX = X'B$$
$$\text{unde lim. } AX = \text{lim. } X'B.$$

Atqui poſito ſemper eſſe $AZ > AX$, $BZ > X'B$; lim. $AX = AZ$, & lim. $X'B = BZ$; proinde $AZ = BZ$.

Exem-

Exemplum ſecundum. Summa quadratorum AZ, BZ debeat eſſe omnium minima.

Sit $AX^2 + BX^2 = AX'^2 + BX'^2$; hinc

$$AX'^2 - AX^2 = BX^2 - BX'^2$$
$$XX'(AX'+AX) = XX'(BX+BX')$$
$$AX' + AX = BX+BX'$$
$$2AX + XX' = 2BX' + XX'$$
$$AX = BX'$$

$\lim.AX = \lim.BX'$; hoc eſt $AZ = BZ$.

Exemplum tertium. Oporteat, ſit $AZ^m \times BZ^n$ = maximum.

Fig. 53. Sit $AX^m \times BX^n = AX'^m \times BX'^n$

Erit $AX^m : AX'^m = BX'^n : BX^n$

$$AX^m : AX'^m - AX^m = BX'^n : BX^n - BX'^n$$

$$AX^m : XX'(AX'^{m-1} + AX'^{m-2}.AX + AX'^{m-3}.AX^2 + \dots + AX'.AX^{m-2} + AX^{m-1})$$
$$= BX'^n : XX'(BX^{n-1} + BX^{n-2}.BX' + BX^{n-3}.BX'^2 + \dots + BX.BX'^{n-2} + BX'^{n-1})$$

unde

$$AX : BX' = \left(\left(\frac{AX'}{AX}\right)^{m-1} + \left(\frac{AX'}{AX}\right)^{m-2} + \dots \frac{AX'}{AX} + 1\right) : \left(\left(\frac{BX}{BX'}\right)^{n-1} + \left(\frac{BX}{BX'}\right)^{n-2} + \dots \frac{BX}{BX'} + 1\right).$$

Quare & limes prioris rationis æqualis eſt limiti rationis poſterioris.

Quoniam autem $\lim.\frac{AX'}{AX} = 1$, & $\lim.\frac{BX'}{BX} = 1$, poſterioris rationis limes eſt $m : n$; & limes rationis prioris eſt $AZ : BZ$: igitur $AZ : BZ = m : n$.

Exemplum quartum. Summa $a \times AZ^m + b \times BZ^m$ debeat eſſe omnium minima.

Sit $a \times AX^m + b \times BX^m = a \times AX'^m + b \times BX'^m$;

erit $b(BX^m - BX'^m) = a(AX'^m - AX^m)$:

unde $b.XX'(BX^{m-1} + BX^{m-2}.BX' + BX^{m-3}.BX'^2 + \dots BX.BX'^{m-2} + BX'^{m-1})$
$= a \times XX'(AX'^{m-1} + AX'^{m-2}.AX + AX'^{m-3}.AX^2 + \dots AX'.AX^{m-2} + AX^{m-1})$

$$b.BX^{m-1}\left(1 + \frac{BX'}{BX} + \left(\frac{BX'}{BX}\right)^2 + \dots \left(\frac{BX'}{BX}\right)^{m-1}\right) = a AX'^{m-1}\left(1 + \frac{AX}{AX'} + \left(\frac{AX}{AX'}\right)^2 + \dots \left(\frac{AX}{AX'}\right)^{m-1}\right)$$

$$b.BX^{m-1} : a.AX'^{m-1} = 1 + \frac{AX}{AX'} + \left(\frac{AX}{AX'}\right)^2 + \dots \left(\frac{AX}{AX'}\right)^{m-1} : 1 + \frac{BX'}{BX} + \left(\frac{BX'}{BX}\right)^2 + \dots \left(\frac{BX'}{BX}\right)^{m-1};$$

unde

unde & prioris rationis limes poſterioris rationis limiti æqualis eſt; nempe $b.BZ^{m-1}:a.AZ^{m-1}=1:1$, ſeu $BZ^{m-1}:AZ^{m-1}=a:b$.

Exemplum quintum. Sint A, B duo puncta poſitione data; & ſit DD' curva ſpecie ac poſitione data. Quæritur hujus curvæ punctum Z, ad quod ductis AZ, BZ rectis, ſit ſumma $a\times AZ^m+b\times BZ^m$ omnium minima. Fig. 54.

Sint X & X' duo puncta ad utramque puncti Z partem ſita, quibus æquales ſummæ propoſitæ reſpondent; ſcilicet $a\times AX^m+b\times BX^m=a\times AX'^m+b\times BX'^m$.

Per Z punctum acta concipiatur recta tangens TT'; centris A, B, radiis AX, BX' deſcribantur arcus Xx', $X'x$, qui rectis AX', BX in x', x occurrant.

Quoniam $a\times AX^m+b\times BX^m=a\times AX'^m+b\times BX'^m$

$$b(BX^m-BX'^m)=a(AX'^m-AX^m):$$

hinc $b.Xx(BX^{m-1}+BX^{m-2}.BX'+BX^{m-3}.BX'^2+\dots BX'^{m-1})$

$=a.X'x'(AX'^{m-1}+AX'^{m-2}.AX+AX'^{m-3}.AX^2+\dots AX^{m-1})$

$$Xx:X'x'=aAX'^{m-1}\left(1+\frac{AX}{AX'}+\frac{AX^2}{AX'^2}+\dots\left(\frac{AX}{AX'}\right)^{m-1}\right):bBX^{m-1}\left(1+\frac{BX'}{BX}+\left(\frac{BX'}{BX}\right)^2+\dots\left(\frac{BX'}{BX}\right)^{m-1}\right);$$

ideoque rationum harum limites pariter ſunt invicem æquales. Atqui X & X' verſus Z punctum ſimul accedentibus eſt

$$\text{lim.}\,Xx : XX' = \text{coſ.}BZT' : 1$$

$$\text{lim.}\,XX' : X'x' = 1 : \text{coſ.}AZT$$

quare $\text{lim.}\,Xx : X'x' = \text{coſ.}BZT' : \text{coſ.}AZT$: unde fit (ut in exemplis antecedentibus) $a.AZ^{m-1} : b.BZ^{m-1} = \text{coſ.}BZT' : \text{coſ.}AZT$,

ſeu $a.AZ^{m-1}\,\text{coſ.}AZT+b.BZ^{m-1}\,\text{coſ.}BZT=0$.

Si plura ſint puncta data, $A, B, C, D, E\dots$ & quæratur ſumma omnium minima $a\times AZ^m+b\times BZ^m+c\times CZ^m+d\times DZ^m+e\times EZ^m\dots$ Erit eodem modo $a.AZ^{m-1}\,\text{coſ.}AZT+b.BZ^{m-1}\,\text{coſ.}BZT+c.CZ^{m-1}\,\text{coſ.}CZT+d.DZ^{m-1}\,\text{coſ.}DZT+e.EZ^{m-1}\,\text{coſ.}EZT+\dots=0$.

Exemplum ſextum. Sit P punctum intra curvam XAX' poſitione datum; per quod agenda ſit recta ZPZ', quæ ſegmentum auferat ZAZ', cujus area ſit omnium minima.

Sint XX', YY' duæ rectæ ad utramque rectæ ZZ' partem ſitæ, quibus æquales areæ XAX', YAY' reſpondeant. Fig. 55.

 Quo-

Quoniam $XAX' = \Upsilon A\Upsilon'$: ſublato ſegmento communi $\Upsilon AX'$, erit $XP\Upsilon = X'P\Upsilon'$. Centro P, radiis $P\Upsilon$, PX' deſcribantur arcus Υx, $X'y'$.

Quoniam $XP\Upsilon = X'P\Upsilon'$, ſeu $XP\Upsilon : X'P\Upsilon' = 1 : 1$

$\lim. XP\Upsilon : \lim. X'P\Upsilon' = 1 : 1$

ſed $\lim. XP\Upsilon : \lim. \Upsilon Px = 1 : 1$

$\lim. \Upsilon Px : X'Py' = PZ^2 : PZ'^2$

$\lim. X'Py' : X'P\Upsilon' = 1 : 1$

ergo $\lim. XP\Upsilon : X'P\Upsilon' = PZ^2 : PZ'^2$.

Proinde area ZPZ' poſita omnium minima, eſt $PZ = PZ'$.

Exemplum ſeptimum. Omnibus ut prius poſitis, recta ZZ' debeat eſſe omnium minima. Iisdem, quæ in exemplo ſexto, factis; per puncta Z, Z' actæ concipiantur rectæ tangentes ZT, $Z'T'$: & ſit $\Upsilon\Upsilon' = XX'$.

Quoniam $XX' = \Upsilon\Upsilon'$: eſt $Xx = \Upsilon'y'$;

et $\lim. Xx : \Upsilon'y' = 1 : 1$

Atqui $\lim. Xx : \Upsilon x = 1 : \text{tang.} PZT$

$\lim. \Upsilon x : X'y' = PZ : PZ'$

$\lim. X'y' : \Upsilon'y' = \text{tang.} PZ'T' : 1$

Ergo $\lim. Xx : \Upsilon'y' = PZ\,\text{tang.} PZ'T' : PZ'\,\text{tang.} PZT$.

Proinde recta ZZ' poſita omnium minima, erit $PZ : PZ' = \text{tang.} PZT : \text{tang.} PZ'T'$.

Exemplum octavum. Arcus ZAZ' debeat eſſe omnium minimus.

Sint arcus XAX', $\Upsilon A\Upsilon'$ invicem æquales: erit ideo $X\Upsilon = X'\Upsilon'$, & $\lim. X\Upsilon : X'\Upsilon' = 1 : 1$.

Atqui $\lim. X\Upsilon : \Upsilon x = 1 : \text{ſin.} PZT$

$\lim. \Upsilon x : X'y' = PZ : PZ'$

$\lim. X'y' : X'\Upsilon' = \text{ſin.} PZ'T' : 1$

Ergo $\lim. X\Upsilon : X'\Upsilon' = PZ\,\text{ſin.} PZ'T' : PZ'\,\text{ſin.} PZT$.

Proinde arcu ZAZ' poſito omnium minimo, eſt $PZ : PZ' = \text{ſin.} PZT : PZ'T'$.

Exem-

Exempla hæc oſtendunt, methodum poſteriorem (quatenus quantitatum, quæ $\frac{\text{maximæ}}{\text{minimæ}}$ reddi debent, evolutionem non requirit) priore methodo generali poſſe eſſe faciliorem ac breviorem. .

§. 185. In exemplis præcedentibus relatio finita inter quantitates incognitas ita fuit poſt differentiationem determinata, ut ipſæ quantitates incognitæ (quoad imperfecta æquationum doctrina permittit) poſſent aſſignari. Transeo ad exempla, quibus quæſtio propoſita indeterminata eſſe videtur, quoniam quantitatum incognitarum numerus videtur æquationum numerum ſuperare: cum tamen quæſtio propoſita reapſe ſit determinata, ſi natura ejus attentius perpendatur. Hoc quæſtionum genus ut facilius intelligatur, nonnulla proponam exempla, a ſimplicioribus ordiendo.

Exemplum primum. Summa propoſita a dividenda ſit in tres partes, ſic ut ſumma quadratorum ex iisdem factorum ſit omnium minima.

Hoc caſu tres occurrunt quantitates incognitæ; & tamen prima fronte duæ tantum proponi conditiones videntur, quarum una ad ſummam datam trium partium refertur, altera ad ſummam omnium minimam quadratorum earundem. Quare problema propoſitum poteſt videri indeterminatum: quod tamen, re accuratius perpenſa, reapſe eſt determinatum. Etenim, parte qualibet eadem manente, duæ reliquæ partes debent eſſe invicem æquales, ut quadratorum ſumma (poſita unius partis conſtantia) ſit omnium minima. Proinde partes propoſitæ binæ ſumtæ debent eſſe invicem æquales; unde tres partes debent eſſe invicem æquales. Ratiocinium hoc applicatur ad numerum quemcunque partium, in quas ſumma data eſt dividenda, ut partium quadratorum ſumma ſit omnium minima. Transeo ad computum.

Sit S ſumma data; & ſint x, y, z partes quæſitæ.

Erit ideo $x + y + z = S$

$xx + yy + zz =$ minimo. Sit v quantitas quædam mutabilis, cujus mutationes pro conſtantibus ſumantur.

Erit

$$\text{Erit} \quad \frac{dx}{dv} + \frac{dy}{dv} + \frac{dz}{dv} = 0$$

$$x\frac{dx}{dv} + y\frac{dy}{dv} + z\frac{dz}{dv} = 0.$$

Hinc poſita z conſtante $\left.\begin{array}{l} \frac{dx}{dv} + \frac{dy}{dv} = 0 \\ x\frac{dx}{dv} + y\frac{dy}{dv} = 0 \end{array}\right\}$; ideoque $x\frac{dx}{dv} - y\frac{dx}{dv} = 0$, & $x = y$.

Eodem modo, poſita y conſtante, $\left.\begin{array}{l} \frac{dx}{dv} + \frac{dz}{dv} = 0 \\ x\frac{dx}{dv} + z\frac{dz}{dv} = 0 \end{array}\right\}$; quare $x\frac{dx}{dv} - z\frac{dx}{dv} = 0$, & $x = z$,

Pariter, poſita x conſtante, $\left.\begin{array}{l} \frac{dy}{dv} + \frac{dz}{dv} = 0 \\ y\frac{dy}{dv} + z\frac{dz}{dv} = 0 \end{array}\right\}$; hinc $y\frac{dy}{dv} - z\frac{dy}{dv} = 0$, & $y = z$.

Unde tres partes quæſitæ debent eſſe invicem æquales.

$$\text{Sit pariter} \quad q + x + y + z = S$$

$$aqq + bxx + cyy + ezz = \text{min.}$$

$$\text{ideoque} \quad \frac{dq}{dv} + \frac{dx}{dv} + \frac{dy}{dv} + \frac{dz}{dv} = 0$$

$$aq\frac{dy}{dv} + bx\frac{dx}{dv} + cy\frac{dy}{dv} + ez\frac{dz}{dv} = 0.$$

Sint y & z conſtantes: erit $\frac{dq}{dv} + \frac{dx}{dv} = 0$

$aq\frac{dq}{dv} + bx\frac{dx}{dv} = 0$: unde $aq = bx$. Eodem modo oſtenditur, debere eſſe $aq = cy = ez$. Proinde $\left.\begin{array}{l} q + x + y + z = S \\ aq = bx = cy = ez \end{array}\right\}$; unde res ad elementa reducitur, & fit $q = S \times \frac{bce}{abc + abe + ace + bce}$.

$$\text{Sit pariter} \quad q^m + x^m + y^m + z^m + \ldots = \text{dato}$$

$$aq^n + bx^n + cy^n + ez^n + \ldots = \text{min.}$$

Erit

Erit $mq^{m-1}\frac{dq}{dv}+mx^{m-1}\frac{dx}{dv}+my^{m-1}\frac{dy}{dv}+mz^{m-1}\frac{dz}{dv}+\ldots=0$

$$anq^{n-1}\frac{dq}{dv}+bnx^{n-1}\frac{dx}{dv}+cny^{n-1}\frac{dy}{dv}+enz^{n-1}\frac{dz}{dv}+\ldots=0:$$

unde, omnibus quantitatibus mutabilibus præter q & x poſitis conſtantibus, fiunt

$$mq^{m-1}\frac{dq}{dv}+mx^{m-1}\frac{dx}{dv}=0;\quad \text{quare}\quad \frac{q^{m-1}}{x^{m-1}}=-\frac{dx}{dq}$$

$$anq^{n-1}\frac{dq}{dv}+bnx^{n-1}\frac{dx}{dv}=0\qquad \frac{aq^{n-1}}{bx^{n-1}}=-\frac{dx}{dq};$$

hinc $aq^{n-m}=bx^{n-m}=cy^{n-m}=ez^{n-m}\ldots.$

ſeu $\sqrt[n-m]{a}\times q=\sqrt[n-m]{b}\times x=\sqrt[n-m]{c}\times y=\sqrt[n-m]{e}\times z\ldots.$

Exemplum ſecundum. Sit $q+x+y+z\ldots=\text{dato}=S$

$$qxyz\ldots\ldots=\text{max.}$$

Igitur $q+x+y+z+\ldots=S$

$$\log.q+\log.x+\log.y+\log.z+\ldots=\text{max.}$$

Hinc $\frac{dq}{dv}+\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}+\ldots=0$

$$\frac{dq}{dv}\cdot\frac{1}{q}+\frac{dx}{dv}\cdot\frac{1}{x}+\frac{dy}{dv}\cdot\frac{1}{y}+\frac{dz}{dv}\cdot\frac{1}{z}+\ldots=0.$$

Poſitis omnibus quantitatibus conſtantibus, præter duas q, x; fiunt

$$\left.\begin{array}{l}\frac{dq}{dv}+\frac{dx}{dv}=0\\ \frac{dq}{dv}\cdot\frac{1}{q}+\frac{dx}{dv}\cdot\frac{1}{x}=0\end{array}\right\};\ \text{unde}\ \frac{1}{q}=\frac{1}{x},\ \text{ſeu}\ q=x:\ \text{quare}\ q=x=y=z\ldots.$$

Si eſſet $q+x+y+z+\ldots=\text{dato}$

$q^m.x^n.y^r.z^s.\ldots=\text{max.}$, ſeu $m\log.q+n\log.x+r\log.y+s\log.z+\cdots=\text{max.}$:

forent $\frac{dq}{dv}+\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}+\ldots.=0$

$m\frac{dq}{dv}\cdot\frac{1}{q}+n\frac{dx}{dv}\cdot\frac{1}{x}+r\frac{dy}{dv}\cdot\frac{1}{y}+s\frac{dz}{dv}\cdot\frac{1}{z}+\ldots.=0.$ Hinc, omnibus quantitatibus præter q & x poſitis conſtantibus, fiunt

$$\frac{dq}{dv} + \frac{dx}{dv} = 0$$

$m\frac{dq}{dv}.\frac{1}{q}+n\frac{dx}{dv}.\frac{1}{x} = 0$: igitur $m.\frac{1}{q}=n.\frac{1}{x}$; & proinde $m.\frac{1}{q}=n.\frac{1}{x}=r.\frac{1}{y}=s.\frac{1}{z}\ldots$

unde res ad calculum vulgarem reducitur per æquationes $\begin{array}{l} q+x+y+z+\ldots=S \\ m.\frac{1}{q}=n.\frac{1}{x}=r.\frac{1}{y}=s.\frac{1}{z}\ldots\ldots \end{array}$

Obſervatio. Hinc determinantur tam $\frac{\text{maxima}}{\text{minima}}$ relativa ſub datis quibusdam conditionibus, quam $\frac{\text{maxima}}{\text{minima}}$ abſoluta, ſeu $\frac{\text{maxima maximorum}}{\text{minima minimorum}}$. Scilicet una pluribusve quantitatibus mutabilibus poſitis conſtantibus, determinantur relationes mutuæ reliquarum, ut $\frac{\text{maximum}}{\text{minimum}}$ ad has conditiones relatum locum habeat: remotis autem his conditionibus, determinatur $\frac{\text{maximum}}{\text{minimum}}$ abſolutum, ſeu $\frac{\text{maximum maximorum}}{\text{minimum minimorum}}$, ſtabiliendo nempe relationes mutuas omnium quantitatum mutabilium, ut $\frac{\text{maximnm maximorum}}{\text{minimum minimorum}}$ locum habeat. Sic uno latere trianguli alicujus magnitudine dato, & ſumma reliquorum magnitudine data; triangulum æquicrurum, in quo latera hæc ſunt invicem æqualia, maximum eſt triangulorum ſub priore conditione conſtructorum: ut vero ſub data perimetro triangulum ſit omnium maximum, ſeu maximum maximorum, tria ejus latera debent eſſe æqualia.

Methodum præcedentem exemplis ad geometriam pertinentibus illuſtrare e re eſſe cenſeo.

§. 186. *Problema.* Inter parallelepipeda rectangula, eadem ſuperficie integra comprehenſa, quæritur illud, cujus capacitas eſt omnium maxima.

Sint x, y, z acies parallelepipedi quæſiti.

Fit ideo $\begin{array}{l} xy+xz+yz = \text{dato} \\ xyz = \text{max.} \end{array}$ ſeu $\begin{array}{l} xy+xz+yz = \text{dato} \\ \log.x+\log.y+\log.z = \text{max.} \end{array}$

hinc $\frac{dx}{dv}(y+z)+\frac{dy}{dv}(x+z)+\frac{dz}{dv}(x+y) = 0$

$$\frac{dx}{dv}.\frac{1}{x} + \frac{dy}{dv}.\frac{1}{y} + \frac{dz}{dv}.\frac{1}{z} = 0.$$

Sit

Sit z conſtans: erit $\frac{dx}{dv}(y+z)+\frac{dy}{dv}(x+z)=0$; quare $y+z:x+z=\frac{1}{x}:\frac{1}{y}=y:x$

$\frac{dx}{dv}\cdot\frac{1}{x} + \frac{dy}{dv}\cdot\frac{1}{y} = 0$ et $z:z = y:x$

unde $y=x$. Eodem modo $y=z$. Proinde $x=y=z$.

Problema. Omnia latera figuræ alicujus rectilineæ præter unum dantur magnitudine. Quæritur figura omnium maxima, lateribus datis & latere non dato comprehenſa.

Lemma. Area figuræ cujusvis rectilineæ ſemiſſis eſt ſummæ rectangulorum omnium laterum binorum ſumtorum, uno excepto, ductorum in ſinus ſummæ angulorum externorum, qui inter latera hæc comprehenduntur. Vid. Opuſculum inſcriptum: *Polygonometrie*, Genève 1791.

Exemplum 1. Figura propoſita ſit quadrilatera.

Sint A, B, C tria latera magnitudine data; & ſint y, z anguli figuræ externi, inter latera A & B, B & C reſpective comprehenſi.

Erit $\left.\begin{array}{l} AB\,\text{ſin.}y \\ +AC\,\text{ſin.}(y+z)+BC\,\text{ſin.}z \end{array}\right\} =$ maximo.

Hinc $\left.\begin{array}{l} AB\frac{dy}{dv}\,\text{coſ.}\,y \\ +AC\left(\frac{dy}{dv}+\frac{dz}{dv}\right)\text{coſ.}(y+z)+BC\frac{dz}{dv}\,\text{coſ.}\,z \end{array}\right\} = 0.$

Ponatur z conſtans: erit $\left.\begin{array}{l} AB\,\text{coſ.}\,y \\ +AC\,\text{coſ.}(y+z) \end{array}\right\} = 0$; unde $B\,\text{coſ.}\,y = C\,\text{coſ.}(180^\circ-(y+z))$

Pariter facto y conſtante $\left.\begin{array}{l} AC\,\text{coſ.}(y+z) \\ +BC\,\text{coſ.}\,z \end{array}\right\} = 0$: unde $B\,\text{coſ.}\,z = A\,\text{coſ.}(180^\circ-(y+z))$.

Exemplum 2. Figura propoſita ſit pentagona.

Sint latera data A, B, C, D;

anguli externi quæſiti x, y, z.

Ideoque (Lemma) $\left.\begin{array}{lll} AB\,\text{ſin.}x & & \\ +AC\,\text{ſin.}(x+y) & +\;BC\,\text{ſin.}y & \\ +AD\,\text{ſin.}(x+y+z) & +\;BD\,\text{ſin.}(y+z) & +\;CD\,\text{ſin.}z \end{array}\right\} =$ max.

 hinc

Hinc $AB\frac{dx}{dv}\text{coſ.}x$

$$+AC\left(\frac{dx}{dv}+\frac{dy}{dv}\right)\text{coſ.}(x+y) \qquad +BC\frac{dy}{dv}\text{coſ.}y \qquad = 0.$$

$$+AD\left(\frac{dx}{dv}+\frac{dy}{dv}+\frac{dz}{dv}\right)\text{coſ.}(x+y+z) +BD\left(\frac{dy}{dv}+\frac{dz}{dv}\right)\text{coſ.}(y+z)+CD\frac{dz}{dv}\text{coſ.}z$$

Poſitis y & z conſtantibus, fit $B\,\text{coſ.}\,x + C\,\text{coſ.}(x+y) + D\,\text{coſ.}(x+y+z) = 0$

poſitis x & z conſtantibus, fit $AC\,\text{coſ.}(x+y)+AD\,\text{coſ.}(x+y+z)+BC\,\text{coſ.}y+BD\,\text{coſ.}(y+z) = 0.$

denique poſitis x & y conſtantibus, fit $A\,\text{coſ.}(x+y+z) + B\,\text{coſ.}(y+z) + C\,\text{coſ.}z = 0$

Proinde quæſtio propoſita ad alteram reducitur mere geometricam (aut algebraicam), qua quantitates quæſitæ x, y, z per tres æquationes datas determinantur.

Inveſtigationis methodum eandem eſſe, quicunque ſit numerus laterum figuræ propoſitæ, ultro patet.

Non immoror demonſtrando formularum harum conſenſui cum altera palmaria figuræ propoſitæ omnium maximæ proprietate, qua ſcilicet figura hæc ſemicirculo inſcribitur, cujus diameter eſt latus quæſitum. (Vide inter alia Opuſcula inſcripta: *De relatione mutua capacitatis & terminorum figurarum*, Varſav. 1782. *Abregé d'Iſoperimetrie elementaire*, Genève 1791.)

Quæri etiam poteſt figura omnium maxima, lateribus magnitudine datis comprehenſa. Inveſtigationis hujus methodum paucis etiam exemplis illuſtrabo; a caſu determinato incipiens, quo figura propoſita eſt quadrilatera.

Sint A, B, C, D latera data quadrilateri, quod omnium maximum fieri debeat. Ducatur recta diagonalis, quæ propoſitum quadrilaterum in duo triangula dividat. Sint A, B crura unius horum triangulorum; & ſit x angulus inter ea comprehenſus. Sint C, D crura alterius trianguli; & ſit y angulus illis comprehenſus.

Erit $AA-2AB\,\text{coſ.}x+BB = CC-2CD\,\text{coſ.}y+DD$

$AB\,\text{ſin.}x+CD\,\text{ſin.}y = \text{maximo.}$

Hinc

$$\text{Hinc } AB\,\text{ſin.}x\frac{dx}{dv} = CD\,\text{ſin.}y\frac{dy}{dv}$$

$$AB\,\text{coſ.}x\frac{dx}{dv}+CD\,\text{coſ.}y\frac{dy}{dv}=0, \quad \text{ſeu } AB\,\text{coſ.}x\frac{dx}{dv}=CD\,\text{coſ.}(180^\circ-y)\frac{dy}{dv}.$$

$$\text{Ergo } AB\,\text{ſin.}x : CD\,\text{ſin.}y = AB\,\text{coſ.}x : CD\,\text{coſ.}(180^\circ-y)$$

$$\text{tang.}x = \text{tang.}(180^\circ-y); \quad x = 180^\circ-y.$$

Maximum igitur quadrilaterum lateribus datis comprehenſum circulo poteſt inſcribi.

Exemplum ſecundum. Quæratur pentagonum, cujus latera A, B, C, D, E dantur magnitudine, & quod ſit omnium maximum.

Ducatur diagonalis, quæ pentagonum propoſitum dividat in triangulum, cujus crura ſint D, E; & in quadrilaterum, cujus latera reliqua ſint A, B, C.

Sit x angulus externus figuræ lateribus A, B interjacens
y - - - - - - B, C - - -
z - - - - - - D, E - - -

Erit (*Polygonometrie*)

$$\left.\begin{matrix} AA+2AB\,\text{coſ.}x \\ +2AC\,\text{coſ.}(x+y)+BB+2BC\,\text{coſ.}y+CC \end{matrix}\right\} = DD+2DE\,\text{coſ.}z+EE$$

$$\left.\begin{matrix} 2AB\,\text{ſin.}x \\ +2AC\,\text{ſin.}(x+y)+BC\,\text{ſin.}y+DE\,\text{ſin.}z \end{matrix}\right\} = \text{maximo.}$$

$$\text{Hinc } \left.\begin{matrix} AB\,\text{ſin.}x\frac{dx}{dv} \\ +AC\,\text{ſin.}(x+y)\left(\frac{dx}{xv}+\frac{dy}{dv}\right) \\ +BC\,\text{ſin.}y\frac{dy}{dv} \end{matrix}\right\} = DE\,\text{ſin.}z\frac{dz}{dv}$$

$$\left.\begin{matrix} AB\,\text{coſ.}x\frac{dx}{dv} \\ +AC\,\text{coſ.}(x+y)\left(\frac{dx}{dv}+\frac{dy}{dv}\right)+BC\,\text{coſ.}y\frac{dy}{dv}+DE\,\text{coſ.}z\frac{dz}{dv} \end{matrix}\right\} = 0.$$

$$\text{Hinc } \frac{dx}{dv}\left(\begin{matrix} AB\,\text{ſin.}x \\ +AC\,\text{ſin.}(x+y)\end{matrix}\right)+\frac{dy}{dv}\left(\begin{matrix} AC\,\text{ſin.}(x+y) \\ +BC\,\text{ſin.}y \end{matrix}\right)-\frac{dz}{dv}DE\,\text{ſin.}z = 0$$

$$\frac{dx}{dv}\left(\begin{matrix} AB\,\text{coſ.}x \\ +AC\,\text{coſ.}(x+y)\end{matrix}\right)+\frac{dy}{dv}\left(\begin{matrix} AC\,\text{coſ.}(x+y) \\ +BC\,\text{coſ.}y \end{matrix}\right)+\frac{dz}{dv}DE\,\text{coſ.}z = 0.$$

 Fiat

Fiat z conſtans: erit

$$B\,\text{ſin.}x + C\,\text{ſin.}(x+y) : B\,\text{coſ.}x + C\,\text{coſ.}(x+y) = A\,\text{ſin.}(x+y) + B\,\text{ſin.}y : A\,\text{coſ.}(x+y) + B\,\text{coſ.}y.$$

Fiat y conſtans: erit

$$B\,\text{ſin.}x + C\,\text{ſin.}(x+y) : B\,\text{coſ.}x + C\,\text{coſ.}(x+y) = \text{ſin.}z : \text{coſ.}(180^\circ - z).$$

Pariter, poſita x conſtante, fit

$A\,\text{ſin.}(x+y) + B\,\text{ſin.}y : A\,\text{coſ.}(x+y) + B\,\text{coſ.}y = \text{ſin.}z : \text{coſ.}(180^\circ - z)$; quæ proportio jam reſultat ex duabus præcedentibus.

Duæ iſtarum proportionum cum æquatione

$$\begin{matrix} AA + 2AB\,\text{coſ.}x \\ + 2AC\,\text{coſ.}(x+y) + BB + 2BC\,\text{coſ.}y + CC \end{matrix} = DD + 2DE\,\text{coſ.}z + EE$$ combinatæ tres exhibent æqnationes, quibus quæſtio propoſita ad problema algebraicum determinatum reducitur.

Tertium exemplum. Quæratur hexagonum lateribus A, B, C, D, E, F magitudine datis comprehenſum, cujus area ſit omnium maxima.

Angulus externus inter latera A, B comprehenſus ſit x

B, C - - - - - y

D, E - - - - - - x'

E, F - - - - - y'.

Erit
$$\begin{matrix} AA + 2AB\,\text{coſ.}x \\ + 2AC\,\text{coſ.}(x+y) + BB + 2BC\,\text{coſ.}y + CC \end{matrix} = \begin{matrix} DD + 2DE\,\text{coſ.}x' \\ + 2DF\,\text{coſ.}(x'+y)' + EE + 2EF\,\text{coſ.}y' + FF \end{matrix}$$

et
$$\begin{matrix} AB\,\text{ſin.}x \\ + AC\,\text{ſin.}(x+y) + BC\,\text{ſin.}y \end{matrix} \begin{matrix} + DE\,\text{ſin.}x' \\ + DF\,\text{ſin.}(x'+y)' + EF\,\text{ſin.}y' \end{matrix} = \text{maximo}.$$

Differentietur utraque æquatio; & omnes quantitates mutabiles x, y, x', y' ponantur ſucceſſive conſtantes, duabus exceptis: hinc eruentur tres relationes diverſæ quantitatum harum mutabilium; quæ cum priori æquatione combinatæ totidem ſuppeditabunt æquationes, quot ſunt quantitates incognitæ.

Theorematibus, quibus formulæ, ad quas pervenitur, anſam præbent, evolvendis non immoror.

Problemata hæc propoſui tanquam exempla univerſalitatis ſolutionum calculo differentiali innixarum: eadem vero aptiſſima ſunt ad oſtendendum, quanto breviores & lucidiores eſſe poſſint ſolutiones mere elementares iis caſibus, in quos

quos quadrant; cum facillime & univerſaliter demonſtretur: figuram lateribus magnitudine datis comprehenſam omnium maximam eam eſſe, quæ circulo poteſt inſcribi. (Vide v. gr. *Opuſcula mea* jam nominata.)

Iisdem principiis inſiſtendo determinari poteſt figura omnium maxima, cujus anguli & perimeter dantur.

Problema. Inter pyramides triangulares æquealtas, baſi ſpecie & magnitudine datæ inſiſtentes, ea quæritur, cujus ſuperficies eſt omnium minima.

Altitudo pyramidis dicatur h; latera baſis ſint A, A', A''. Anguli (quæſiti), ſub quibus facies pyramidis lateribus his reſpective adjacentes ad baſin inclinantur, ſint x, x', x'' reſpective. Puncti, in quo altitudo pyramidis baſi occurrit, a lateribus baſis diſtantiæ ſunt reſpective $h\cot.x$, $h\cot.x'$, $h\cot.x''$.

Altitudines facierum pyramidis ſunt $h\,\text{coſec.}x$, $h\,\text{coſec.}x'$, $h\,\text{coſec.}x''$.

Duplum areæ baſis eſt $h(A\cot.x + A'\cot.x' + A''\cot.x'')$.

Dupla ſumma facierum eſt $h(A\,\text{coſec.}x + A'\,\text{coſec.}x' + A''\,\text{coſec.}x'')$.

Ideoque
$$A\cot.x + A'\cot.x' + A''\cot.x'' = \text{dato}$$
$$A\,\text{coſec.}x + A'\,\text{coſec.}x' + A''\,\text{coſec.}x'' = \text{minimo.}$$

Hinc
$$A\frac{dx}{dv}\,\text{coſec.}^2x + A'\frac{dx'}{dv}\,\text{coſec.}^2x' + A''\frac{dx''}{dv}\,\text{coſec.}^2x'' = 0$$
$$A\frac{dx}{dv}\cot.x\,\text{coſec.}x + A'\frac{dx'}{dv}\cot.x'\,\text{coſec.}x' + A''\frac{dx''}{dv}\cot.x''\,\text{coſec.}x'' = 0.$$

Fiat x'' conſtans: erit
$$A\frac{dx}{dv}\,\text{coſec.}^2x + A'\frac{dx'}{dv}\,\text{coſec.}^2x' = 0$$
$$A\frac{dx}{dv}\cot.x\,\text{coſec.}x + A'\frac{dx'}{dv}\cot.x'\,\text{coſec.}x' = 0.$$

Hinc
$$A\frac{dx}{dv} : -A'\frac{dx'}{dv} = \text{coſec.}^2x' : \text{coſec.}^2x$$
$$A\frac{dx}{dv} : -A'\frac{dx'}{dv} = \cot.x'\,\text{coſec.}x' : \cot.x\,\text{coſec.}x$$

Proinde
$$\text{coſec.}^2x' : \text{coſec.}^2x = \cot.x'\,\text{coſec.}x' : \cot.x\,\text{coſec.}x$$
$$\text{coſec.}x' : \text{coſec.}x = \cot.x' : \cot.x$$
$$1 : 1 = \text{coſ.}x' : \text{coſ.}x;$$ ideoque $x = x'$.

Eodem modo infertur $x = x''$. Pyramidum igitur triangularium æquealtarum,

baſi

baſi datæ inſiſtentium, ea terminatur ſuperficie minima, cujus facies ad baſin æqualiter inclinantur. Vide Opuſcula modo commemorata, in quibus propoſitio hæc methodo mere elementari adſtruitur, & uberes ejus conſequentiæ evolvuntur.

§. 187. In quæſtionibus poſtremo loco tractatis quantitas, quæ $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$ reddi debet, functio eſt plurium quantitatum mutabilium, a ſe invicem ita independentium, ut determinatio unius reliquas non determinet. Criteria, quibus dijudicari poteſt, utrum ejusmodi functio $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$ eſſe poſſit, nec ne? primus omnium univerſaliſſime conſtituit celeb. DE LA GRANGE (*Miſcellanea Societatis Taurinenſis*, Tom. I.). Sufficiat hoc loco, functiones duarum variabilium conſiderare, atque criteria hæc ex formulis capite præcedenti traditis deducere.

Quoniam eſt

$$\overset{x}{{}^{y}P} = P \begin{array}{llll} + \frac{\Delta x}{1}{}^{x}d'P + & \frac{\Delta x^2}{1.2}{}^{x}d''P & + \frac{\Delta x^3}{1.2.3}{}^{x}d'''P & \\ + \frac{\Delta y}{1}{}^{y}d'P + & 2\frac{\Delta x.\Delta y}{1.2}{}^{y}d'{}^{x}d'P + & 3\frac{\Delta x^2.\Delta y}{1.2.3}{}^{y}d{}^{x}d''P & \\ & + \frac{\Delta y^2}{1.2}{}^{y}d''P & + 3\frac{\Delta x.\Delta y^2}{1.2.3}{}^{y}d''{}^{x}d'P & + \ldots \\ & & + \frac{\Delta y^3}{1.2.3}{}^{y}d'''P & \end{array}$$

ut functio hæc fiat $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$, debet eſſe $\begin{matrix}{}^{x}d'P = 0\\ {}^{y}d'P = 0\end{matrix}$

Eſt autem

$$\begin{array}{l}\Delta x^2\,{}^{x}d''P \\ +2\Delta x.\Delta y\,{}^{y}d'{}^{x}d'P \\ +\ \Delta y^2\,{}^{y}d''P\end{array} = {}^{x}d''P\left[\Delta x^2 + 2\Delta x.\Delta y\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P} + \Delta y^2\frac{{}^{y}d''P}{{}^{x}d''P}\right]$$

$$= {}^{x}d''P\left[\Delta x^2 + 2\Delta x.\Delta y\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P} + \Delta y^2\left[\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2\right]$$

$$-\Delta y^2.{}^{x}d''P\left[\left[\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2 - \frac{{}^{y}d''P}{{}^{x}d''P}\right]$$

$$= {}^{x}d''P\left[\Delta x + \Delta y\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2 + \Delta y^2\,{}^{x}d''P\left[\frac{{}^{y}d''P}{{}^{x}d''P} - \left[\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2\right]$$

$$= {}^{x}d''P\left[\Delta x + \Delta y\frac{{}^{y}d'{}^{x}d'P}{{}^{x}d''P}\right]^2 + \Delta y^2\left[{}^{y}d''P - \frac{({}^{y}d'{}^{x}d'P)^2}{{}^{x}d''P}\right].$$

Pro-

Proinde quoniam quadrata $\left[\Delta x + \Delta y \frac{{}^{y}d'\,{}^{x}d'P}{{}^{x}d''P}\right]^2$, Δy^2 ſemper ſunt poſitiva, functio P maxima eſt, ſi ſit tam ${}^{x}d''P$, quam ${}^{y}d''P - \frac{({}^{y}d'\,{}^{x}d'P)^2}{{}^{x}d''P}$ negativa: & poſita ${}^{x}d''P$ negativa, ac proinde $-\frac{({}^{y}d'\,{}^{x}d'P)^2}{{}^{x}d''P}$ poſitiva; neceſſe eſt, ut ſit ${}^{y}d''P$ negativa; prætereaque (ſignis omiſſis) debet eſſe ${}^{y}d''P > \frac{({}^{y}d'\,{}^{x}d'P)^2}{{}^{x}d''P}$, ſeu ${}^{x}d''P \times {}^{y}d''P > ({}^{y}d'\,{}^{x}d'P)^2$.

Contra functio P minima fit, ſi tam ${}^{x}d''P$, quam ${}^{y}d''P - \frac{({}^{y}d'\,{}^{x}d'P)^2}{{}^{x}d''P}$ eſt poſitiva: & poſita ${}^{x}d''P$ poſitiva; oportet, ſit ${}^{y}d''P$ poſitiva, eaque talis, ut ${}^{y}d''P.\,{}^{x}d''P > ({}^{y}d'\,{}^{x}d'P)^2$.

Exemplum hoc functionis duarum variabilium docet: criteria, quibus $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$ functionum plurium variabilium diſcernuntur, nonniſi ſumma adhibita cautione poſſe determinari; niſi ex ipſa quæſtionis natura immmediate judicari poſſit, utrum quæſtio ad $\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}$ referenda ſit, aut nulli limiti anſam præbeat?

Exemplum primum. Sit

$$\begin{array}{llll} P = xx + xy + yy - ax - by & {}^{x}d''P = +2 & & \\ {}^{x}d'P = 2x + y \quad - a & & {}^{y}d'\,{}^{x}d'P = 1; \\ {}^{y}d'P = \quad x + 2y \quad - b & {}^{y}d''P = +2 & \end{array}$$

proinde ${}^{x}d''P.\,{}^{y}d''P (= 4) > ({}^{y}d'\,{}^{x}d'P)^2$.

Fiat $\begin{matrix}{}^{x}d'P (= 2x + y - a) = 0 \\ {}^{y}d'P (= x + 2y - b) = 0\end{matrix}$. Erit $\begin{matrix} x = \frac{2a - b}{3} \\ y = \frac{2b - a}{3}\end{matrix}$; & functio P his valoribus reſpondens, nempe $-\frac{aa - ab + bb}{3}$, eſt omnium minima. (a)

Exem-

(a) Irrepſit error typographicus in pag. 649. Calculi differentialis Euleri, ubi functio hæc minima dicitur eſſe $-\frac{aa + ab - bb}{3}$, loco $\frac{-aa + ab - bb}{3}$.

Exemplum ſecundum. Sit $P = x^3 - 3axy + y^3 \qquad {}^{y}\mathrm{d}'P = -3ax + 3yy$

$${}^{x}\mathrm{d}'P = 3xx - 3ay \qquad\qquad {}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P = -3a.$$

$${}^{x}\mathrm{d}''P = 2.3x \qquad {}^{y}\mathrm{d}''P = +2.3y$$

Fiat $\begin{matrix} xx - ay = 0 \\ -ax + yy = 0 \end{matrix}$: erit $\begin{matrix} y = \dfrac{xx}{a} \\ -ax + \dfrac{x^4}{aa} = 0; \quad x = \begin{matrix}0\\a\end{matrix} \quad y = \begin{matrix}0\\a\end{matrix}. \end{matrix}$

Quoniam factis $\begin{matrix} x=0 \\ y=0 \end{matrix}$, eſt ${}^{x}\mathrm{d}''P \times {}^{y}\mathrm{d}''P < ({}^{y}\mathrm{d}'\,{}^{x}\mathrm{d}'P)^2$; functio P his valoribus reſpondens non eſt omnium $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$.

Fiant $\begin{matrix} x=a \\ y=a \end{matrix}$: erunt $\begin{matrix} {}^{x}\mathrm{d}''P = 6a \\ {}^{y}\mathrm{d}''P = 6a \end{matrix}$; proinde ${}^{x}\mathrm{d}''P . {}^{y}\mathrm{d}''P > ({}^{y}\mathrm{d}'P . {}^{x}\mathrm{d}'P)^2$, & functio fit $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$. Atqui $\begin{matrix} {}^{x}\mathrm{d}''P = +6a \\ {}^{y}\mathrm{d}''P = +6a \end{matrix}$; igitur functio $P = -a^3$ minima eſt.

§. 188. Quæſtiones ad $\begin{matrix}\text{maxima}\\\text{minima}\end{matrix}$ pertinentes magnam cum ductu tangentium a puncto extra curvam dato habent affinitatem.

Per punctum enim datum agatur recta, ad quam tanquam axem curva referatur. Si curva verſus axem hunc concava eſt; angulus, quem recta curvam tangens ab hoc puncto ducta facit cum axe, major eſt angulis, quos rectæ curvam ſecantes ab eodem puncto ductæ faciunt cum eodem axe: contra prior angulus minor eſt poſterioribus, ſi curva eſt verſus axem convexa. Problema igitur, a puncto extra curvam dato rectam ducere, quæ curvam contingat, ad alterum reducitur, quo ab eodem puncto recta duci jubetur curvæ occurrens, quæ $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$ angulum cum axe comprehendat. Subtangens curvæ dicatur t; ordinata rectangula per punctum contactus acta dicatur y: erit $\frac{y}{t}$ tangens trigonometrica anguli, quem recta curvam contingens cum axe facit; proinde $\frac{y}{t}$ $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$: unde $\dfrac{t\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}t}{\mathrm{d}x}}{tt} = 0$; $t\dfrac{\mathrm{d}y}{\mathrm{d}x} = y\dfrac{\mathrm{d}t}{\mathrm{d}x}$. Atqui $\dfrac{\mathrm{d}t}{\mathrm{d}x} = 1$, ſive abſciſſarum origo in ipſo puncto dato ponatur, ſive in alio puncto intervallo dato a priori diſtante. Ergo $t \times \dfrac{\mathrm{d}y}{\mathrm{d}x} = y$, & $t = y \times \dfrac{\mathrm{d}x}{\mathrm{d}y}$; quod conſentit cum §. 40.

Rela-

Relatio hæc non effugit ſuperioris ſeculi mathematicos, quorum molimina utrumque problema de ductu tangentium & de maximis ac minimis ſolvendi ad generales calculi differentialis regulas viam ſtraverunt; quos inter FERMATIUM, ROBERVALLIUM, PASCALIUM, BARROWIUM nominare ſufficiat.

§. 189. Quamvis quæſtiones ad $\frac{\text{maxima}}{\text{minima}}$ pertinentes inde ab eo tantum tempore cenſeri debeant perfecte ſolutæ, quo generales calculi differentialis regulæ iis applicatæ fuerunt: abunde tamen, quæ ſuperſunt, monumenta docent, veteres geometras hujus generis quæſtionibus operam haud inanem impendiſſe. Ipſa Elementa EUCLIDIS propoſitiones nonnullas ad $\frac{\text{maxima}}{\text{minima}}$ pertinentes ſiſtunt, ex. gr. in 5ta & 9na Libri ſecundi. Luculentiſſime autem hoc comprobant libri antiquorum ad analyſin geometricam pertinentes, a PAPPO recenſi, & partim ſuperſtites, partim a recentioribus mathematicis reſtituti; præſertim Tractatus *de Sectione rationis & ſpatii*, *de Sectione determinata*, & *de Inclinationibus*, atque inprimis Liber V. *Sectionum conicarum* APOLLONII.

Quando problema aliquod geometrice ſolvitur; defectus occurſus mutui linearum in conſtructione ducendarum monet de propoſitæ quæſtionis impoſſibilitate: algebra autem impoſſibilitatem hanc indicat per introductionem quantitatum (quas vocant) imaginariarum, ſeſe invicem non deſtruentium. Sed imperfecta ſolutionis æquationum conditio obſtat, quominus algebra hoc reſpectu ſufficiat. Cum vero æquationum ſecundi tantum gradus ſolutio plana ſit; functionum ejusdem ordinis $\frac{\text{maximæ}}{\text{minimæ}}$ facillime per algebram elementarem determinantur, quod paucis exemplis declarabo.

Primum exemplum. Sit $2ax-xx=x(2a-x)$ functio ſecundi ordinis, cujus maximum quæritur. Sit $2ax-xx=p$: erit $x-a=\pm\sqrt{(aa-p)}$. Ut problema ſit poſſibile; requiritur, non ſit p major quam aa: proinde maximus ipſius p valor eſt aa; & tunc $x-a=0$, $x=a$, $2a-x=a$; ſeu duo termini producti $x(2a-x)$ ſunt invicem æquales.

Exemplum ſecundum. Sit numerus datus a dividendus in duas partes, ſic ut ſumma productorum quadratorum ipſarum per numeros datos m, n ſit omnium minima. Sint duæ partes x, $a-x$; & ponatur $mxx+n(a-x)^2=p$:

erit

erit $x - aa\frac{n}{m+n} = \frac{\pm \sqrt{(p(m+n) - aamn)}}{m+n}$. Ut problema poſſibile ſit, debet eſſe $p(m+n) \overline{\geq} aamn$; ſeu minimus ipſius p valor eſt $aa\frac{mn}{m+n}$: & caſu, quo p eſt omnium minima, fit $x = \frac{an}{m+n}$, $a - x = \frac{am}{m+n}$.

Exemplum tertium. Sit $\sqrt{(aa+xx)} - \frac{m}{n}x = b$.

$$\text{unde} \quad aa + xx = bb + \frac{2m}{n}bx + \frac{mm}{nn}xx.$$

1°. Sit $m = n$: prodit æquatio primi gradus, quæ $\frac{\text{maximo}}{\text{minimo}}$ locum non præbet.

2°. Sit $m > n$: $x + b\frac{mn}{mm - nn} = \pm \sqrt{\left(aa\frac{nn}{mm - nn} + bb\frac{n^4}{(mm - nn)^2}\right)}$; unde functio propoſita nullum admittit limitem.

3°. Sit $m < n$: $x - b\frac{mn}{nn - mm} = \pm \sqrt{\left(bb\frac{n^4}{(nn - mm)^2} - aa\frac{nn}{nn - mm}\right)}$. Itaque, poſito $m < n$, valor omnium minimus ipſius b talis eſt, ut ſit

$$b = a\frac{\sqrt{(nn - mm)}}{n};$$

$$\text{\& tum eſt} \quad x = b \times \frac{mn}{nn - mm} = a \times \frac{m}{\sqrt{(nn - mm)}}$$

$$\sqrt{(aa+xx)} = a \times \frac{n}{\sqrt{(nn - mm)}}.$$

Methodum hanc elementarem inveſtigationi minimi ceræ apium cellularum feliciſſime applicuit ſagaciſſimus Dn. Le Sage; qui ita celebre hoc tam apud mathematicos quam apud rerum naturalium ſtudioſos problema primus ad elementa reduxit, & quidem ad inveſtigationem minimi functionis $\sqrt{(aa+xx)} - \frac{m}{n}x$ caſu, quo $m < n$ (uti in poſteriori exemplo). Solutionem hanc amiciſſime mecum communicavit, quo tempore ſtudia mea mathematica paterno amore regebat; pariter ac plurimas obſervationes ad hoc caput pertinentes, quæ extant in *Commentariis Academiæ Berolinenſis* ad annum 1782: *Sur le minimum de cire des alveoles des abeilles, & en particulier ſur un minimum minimorum relatif à cette matiere.* Vid. etiam *Relatio mutua &c.*

CAPUT

Caput decimum nonum.

De radiis curvaturæ, et de curvis evolutione genitis.

§. 190.

Sint M, M' duo puncta alicujus curvæ; per quæ agantur duæ rectæ normales MN, $M'N'$, quæ axi in punctis N, N' occurrant. Diſtantia NN' punctorum horum eſt $PP - PN + P'N' = \Delta x - y\frac{dy}{dx} + y'\frac{dy'}{dx}$.

Quoniam

$$y'y' = yy + 2\frac{\Delta x}{1}y\frac{dy}{dx} + 2\frac{\Delta x^2}{1.2}\left[\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + 2\frac{\Delta x^3}{1.2.3}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + 2\frac{\Delta x^4}{1\ldots4}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \ldots$$

ſit

$$y'\frac{dy'}{dx} = y\frac{dy}{dx} + \frac{\Delta x}{1}\left[\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + \frac{\Delta x^2}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + \frac{\Delta x^3}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \frac{\Delta x^4}{1\ldots4}\left[10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right] + \ldots$$

Unde

$$NN' = \Delta x\left[1 + \left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2}\right] + \frac{\Delta x^2}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] + \frac{\Delta x^3}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] + \frac{\Delta x^4}{1\ldots4}\left[10\frac{ddy}{dx^2}\cdot\frac{d^3y}{dx^3} + 5\frac{dy}{dx}\cdot\frac{d^4y}{dx^4} + y\frac{d^5y}{dx^5}\right] + \ldots$$

§. 191. Duæ normales MN, $M'N'$ ſibi invicem (ſi fieri poſſit) occurrant in Z puncto. Punctum hoc occurſus ſic determinatur.

In

Fig. 56. In triangulo NZN' eſt ſin.NZN' : ſin.N' $= NN' : NZ$

$$\text{ſeu}\quad \text{ſin.}(M'N'P' - MNP) : \text{ſin.}M'N'P' = NN' : NZ$$

$$\text{unde}\quad \text{coſ.}N - \text{ſin.}N\text{cot.}N' : 1 = NN' : NZ$$

$$\text{coſ.}N - \text{ſin.}\frac{dy'}{dx} : 1 = NN' : NZ$$

$$NP - MP\frac{dy'}{dx} : MN = NN' : NZ$$

$$\text{et}\quad NP - MP\frac{dy'}{dx} : NN' = MN : NZ$$

$$\text{Atqui}\quad \frac{dy'}{dx} = \frac{dy}{dx} + \frac{\Delta x}{1}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^3}{1.2.3}\cdot\frac{d^4y}{dx^4} + \ldots.$$

$$\text{Igitur}\quad NP - MP\frac{dy'}{dx} = -y\left(\frac{\Delta x}{1}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^2}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^3}{1.2.3}\cdot\frac{d^4y}{dx^4} + \ldots\right)$$

$$\begin{aligned} NZ : MN = {} & 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2} \qquad : -y\left[\frac{ddy}{dx^2} + \frac{\Delta x}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^4y}{dx^4} + \ldots\right] \\ & + \frac{\Delta x}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] \\ & + \frac{\Delta x^2}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] \\ & + - \quad - \quad - \quad - \quad - \quad - \\ & + - \quad - \quad - \quad - \quad - \quad - \end{aligned}$$

$$\begin{aligned} \text{et}\quad MZ : MN = {} & 1 + \left(\frac{dy}{dx}\right)^2 \qquad : -y\left[\frac{ddy}{dx^2} + \frac{\Delta x}{1.2}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^4y}{dx^4} + \ldots\right] \\ & + \frac{\Delta x}{1.2}\left[3\frac{dy}{dx}\cdot\frac{ddy}{dx^2} + y\frac{d^3y}{dx^3}\right] \\ & + \frac{\Delta x^2}{1.2.3}\left[3\left(\frac{ddy}{dx^2}\right)^2 + 4\frac{dy}{dx}\cdot\frac{d^3y}{dx^3} + y\frac{d^4y}{dx^4}\right] \\ & + - \quad - \quad - \quad - \quad - \quad - \\ & + - \quad - \quad - \quad - \quad - \quad - \end{aligned}$$

Ut conſequentiæ ex hac proportione necti poſſint, duo diſtinguendi ſunt caſus, uti in §. 170. 179.

§. 192. *Primus caſus.* Exponentes differentiales ſucceſſivi $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, . . . ſigno impoſſibilis non afficiantur.

Caſu hoc poſterioris rationis limes æqualis eſt rationi $1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$; proinde & rationis prioris limes æqualis eſt eidem rationi $1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$, ſeu $\lim. MZ : MN = 1+\left(\frac{dy}{dx}\right)^2 : -y\frac{ddy}{dx^2}$, & $\lim. MZ = MN \times \frac{1+\left(\frac{dy}{dx}\right)^2}{-y\frac{ddy}{dx^2}}$. Atqui $MN = y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$ (§. 110.) Ergo $\lim. MZ = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$.

Obſervatio prima. Quamvis hæc expreſſio formam habeat negativam, reapſe ſit poſitiva, quando curva verſus axem concava eſt.

Obſervatio ſecunda. Ex Z puncto agatur ZQ ipſi MP perpendicularis. Eſt $MQ = MZ \times \frac{MP}{MN} = MZ \times \frac{1}{\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$: unde $\lim. MQ = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$.

Obſervatio tertia. Quoniam $\text{ſin.} NMP = \frac{\frac{dy}{dx}}{\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}}$; eſt $\frac{d.\,\text{ſin.} NMP}{dx} = \frac{\frac{ddy}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$. Hinc $\lim. \frac{1}{MZ} = -\frac{d.\,\text{ſin.} NMP}{dx} = -\frac{d.\,\text{coſ.} PMT}{dx}$.

Æquatio hæc immediate ſic poteſt demonſtrari. Recta $M'P'$ ipſi ZQ occurrente in p, ſunt $\text{ſin.} NMP = \frac{ZQ}{ZM}$, $\text{ſin.} N'M'P' = \frac{Zp}{ZM'}$: hinc $\text{ſin.} NMP - \text{ſin.} N'M'P'$ $= \frac{ZQ}{ZM} - \frac{Zp}{ZM'} = \frac{PP'}{ZM} + Zp\left(\frac{1}{ZM} - \frac{1}{ZM'}\right)$; $-\frac{\Delta.\text{ſin.} NMP}{\Delta x} = \frac{1}{ZM} + \frac{Zp}{\Delta x} \times \frac{ZM' - ZM}{ZM \cdot ZM'}$, & $-\frac{d.\text{ſin.} NMP}{dx} = \frac{1}{ZM}$.

Obſer-

Obſervatio quarta. $M'P'$ recta occurrat in t tangenti MT per M punctum ductæ. $M't = -\left(\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4}+\ldots\right)$ (§. 170.)

et $Mt^2 = \Delta x^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$.

Proinde $\frac{Mt^2}{M't} = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\left(\frac{1}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x}{1.2.3}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^2}{1\ldots4}\cdot\frac{d^4y}{dx^4}+\ldots\right)}$

et $\lim.\frac{Mt^2}{M't} = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{1}{1.2}\cdot\frac{ddy}{dx^2}} = 2\,\frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$.

Atqui $\lim.MQ = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$. Ergo $2\lim.MQ = \lim.\frac{Mt^2}{M't}$.

§. 193. Sumta $Mq = 2MQ = \lim.\frac{Mt^2}{M't} = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}$, fiat ſemper $tq' = \frac{Mt^2}{M't}$; & deſcribatur curva per puncta q, q' hoc modo determinata transiens. Tum centro Z radio $MZ = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$ deſcribatur circumferentia circuli; quæ proinde per M & q transit, atque rectam MT, ac proinde etiam curvam MM' in M tangit. Circumferentia hæc occurrat rectæ $M'P'$ in R & R' punctis.

Fig. 56. 1°. *Primus caſus.* Sit Mq quantitatis $\frac{Mt^2}{M't}$ limes quod ad parvitatem; ideoque ſemper ſit $tq' > tR'$, & arcus qq' extra arcum qR'. Quoniam $Mt^2 = R't\times tR = M't\times tq'$, & $tq' > tR'$; ergo $tR > M't$: proinde arcus circuli MR cadit intra arcum curvæ MM'; ac tanto magis quælibet circumferentia rectam MT in M tangens & radio minore quam MZ deſcripta, utpote circulum ipſum MR intus tangens, intra curvam MM' cadit. Tum radio MZ' majore quam MZ deſcribatur circulus, qui rectæ MP in $'q$ puncto, & rectæ $M'P'$ in r & r' punctis

ctis occurrat. Erit ideo $M'q > Mq$, & punctum $'q$ extra curvam qq'; pariterque arcus $'qr'$ extra arcum qR': quare, puncto M' ad punctum M accedente, arcus curvæ qq' cadit inter arcus circulares qR', $'qr'$; itaque erit $tq' < tr'$. Atqui $Mt^2 = M't \times tq' = rt \times tr'$: igitur $M't > rt$; & circumferentia radio $Z'M > ZM$ deſcripta cadit inter arcum MM' & tangentem MT. Proinde hoc caſu arcus curvæ MM' jacet inter arcus circulares MR, Mr, ſic ut nullus circulus, curvam in puncto M tangens, inter arcus MM' & MR cadat.

Secundus caſus. Sit Mq quantitatis $\frac{Mt^2}{M't}$ limes quod ad magnitudinem; ita ut ſemper ſit $tq' < tR'$, & arcus qq' intra arcum qR'. Quoniam $Mt^2 = Rt \times tR' = M't \times tq'$, & $tq' < tR'$: erit $Rt < M't$; proinde arcus MR cadit inter tangentem & arcum MM'; ac tanto magis quælibet circumferentia radio majore quam MZ deſcripta, quippe circulum MR extus contingens, inter tangentem MT & arcum MM' cadit. Tum radio $MZ' < MZ$ deſcribatur circulus, qui rectæ MP' in r & r' punctis, & rectæ MQ in $'q$ puncto occurrat. Erit ideo $M'q < Mq$, $tr' < tR'$, & arcus qq' jacebit inter arcus qR', $'qr'$. Proinde puncto M' ad punctum M accedente, fiet $tq' > tr'$; ideoque $tM' < tr$, & arcus MM' cadet inter tangentem MT & arcum Mr. Quare hoc etiam caſu nullus circulus, curvam in M tangens, inter arcus MM' & MR tranſit. Fig. 56. 2°.

Nullus itaque circulus curvam in M tangens, radio ſive majore ſive minore quam MZ deſcriptus, cadit inter arcum curvæ MM' & arcum circularem MR, ad partes puncto M utrinque vicinas. Circumferentia igitur radio MZ deſcripta arcum curvæ MM' ſtrictius tangit, quam ulla alia circumferentia, ſive major ſive minor priore. Hinc circumferentia hæc curvam oſculari, & radius ZM *radius oſculator* dicitur. Curvatura arcus MM' ad curvaturam arcus circularis MR propius accedit, quam ad curvaturam cujusvis alius arcus circularis; unde radius ZM vocatur *radius curvaturæ* arcus MM' in puncto M.

Formula ideo radii curvaturæ eſt $\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$; cujus applicationem nonnullis exemplis illuſtrabo.

§. 194. *Exempla.* 1°. Sit parabola conica, cujus æquatio eſt $yy = 2px$. Fit $y\frac{dy}{dx} = p$, $y\frac{ddy}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$, $y\frac{ddy}{dx^2} = -\frac{pp}{yy}$, $\frac{ddy}{dx^2} = -\frac{pp}{y^3}$, $1 + \left(\frac{dy}{dx}\right)^2 = \frac{pp+yy}{yy}$. $MZ = \frac{(pp+yy)^{\frac{3}{2}}}{pp}$, & $MQ = \frac{y(pp+yy)}{pp}$.

Obſervatio. Facto $y = 0$, fit $MZ = p$; & hic eſt limes (quod ad parvitatem) normalium parabolæ.

2°. Sit ellipſis conica, cujus æquatio eſt $yy = \frac{bb}{aa}(aa - xx)$. Hinc $y\frac{dy}{dx} = -\frac{bb}{aa}x$, $\left(\frac{dy}{dx}\right)^2 + y\frac{ddy}{dx^2} = -\frac{bb}{aa}$, $y\frac{ddy}{dx^2} = -\frac{b^4}{aa}\cdot\frac{1}{yy}$, $\frac{ddy}{dx^2} = -\frac{b^4}{aa}\cdot\frac{1}{y^3}$. $MZ = \frac{((aa-bb)yy+b^4)^{\frac{3}{2}}}{ab^4}$. Fiat $y = 0$: erit $MZ = \frac{bb}{a}$; & hic eſt limes (quod ad parvitatem) normalium ellipſis. Sit $a = b$: erit $MZ = \frac{a^6}{a^5} = a$, ſeu conſtans; quo caſu ellipſis in circulum abiit.

3°. Eodem modo determinatur radius curvaturæ hyperbolæ conicæ ex æquatione $yy = \frac{bb}{aa}(xx - aa)$; qua fit $MZ = \frac{((aa+bb)yy + b^4))^{\frac{3}{2}}}{ab^4}$.

4°. Sit cyclois vulgaris, cujus æquatio eſt $y = \sqrt{(2rx - xx)} + r\,\text{arc.ſin.v.}\frac{x}{r}$. Fit $\frac{dy}{dx} = \sqrt{\frac{2r-x}{x}}$, $\frac{ddy}{dx^2} = -\frac{r}{x\sqrt{(2rx-xx)}}$, $1 + \left(\frac{dy}{dx}\right)^2 = \frac{2r}{x}$; $MZ = 2r\sqrt{(2r(2r-x))}$.

§. 195. Hucusque tradita nituntur ſuppoſitione, dari limitem quoti $\frac{Mt^2}{M't}$; quo caſu radius curvaturæ poteſt determinari. Ut *alterum caſum*, quo nullus eſt quoti hujus limes, diſtinctius evolvam; a curvis ordiar, quarum æquatio ſimpliciſſima eſt, nempe $y = p^{1-n}x^n$.

Deſcribatur quicunque circulus curvam in vertice contingens, cujus radius ſit r; & rectæ in hoc circulo axi ordinatim applicatæ ſint y'.

Quoniam $y = p^{1-n}x^n$, eſt $yy = p^{2-2n}x^{2n}$.

Atqui $y'y' = x(2r - x)$

Ergo $yy : y'y' = p^{2-2n}x^{2n} : x(2r - x)$.

1°. Sit

1°. Sit $2n = 1$, ſeu $n = \frac{1}{2}$: erit $yy : y'y' = p^{2-2n} : 2r - x = p : 2r - x$. Proinde, imminuta x, rationis poſterioris limes eſt ratio $p : 2r$; & ratio hæc eſt ratio æqualitatis, ſi $p = 2r$, ſeu $r = \frac{1}{2}p$.

2°. Sit $2n > 1$; ideoque $yy : y'y' = p^{2-2n}x^{2n-1} : 2r - x$. Imminuta x, nullus eſt limes (quod ad parvitatem) poſterioris rationis; itaque nullus etiam eſt limes (quod ad parvitatem) rationis prioris: quælibet igitur circuli circumferentia, curvam in vertice contingens, cadit extra curvam ad partes vertici vicinas, utcunque parvus ſit radius ejus r.

3°. Sit $2n < 1$; $yy : y'y' = p^{2-2n} : x^{1-2n}(2r - x)$. Imminuta x, nullus eſt poſterioris rationis limes, quod ad magnitudinem; ideoque nullus etiam eſt prioris rationis limes, quod ad magnitudinem. Proinde quælibet circuli circumferentia curvam in vertice contingens cadit intra curvam ad partes vertici vicinas, utut magnus ſit ejus radius r.

Ideoque parabolas inter, æquatione $y = p^{1-n}x^n$ determinatas, parabola conica ſola eſt, cujus curvatura in vertice cum curvatura circuli conferri poſſit.

Idem conſequitur ex applicatione formulæ generalis radii oſculi ad caſum, in quem quadrare non poteſt: eum nempe, quo nullus eſt quoti $\frac{Mt^2}{M't}$ limes, ſive quod ad magnitudinem, ſive quod ad parvitatem; ſeu quo formula radii curvaturæ $R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{-\frac{ddy}{dx^2}}$ ſit aut zero, aut impoſſibilis.

Etenim poſito $y = p^{1-n}x^n$, fit $\frac{dy}{dx} = np^{1-n}x^{n-1}$

$$\frac{ddy}{dx^2} = n \cdot \overline{n-1}\, p^{1-n}x^{n-2}$$

$$\text{unde} \quad R = \frac{(1 + nnp^{2-2n}x^{2n-2})^{\frac{3}{2}}}{-n \cdot \overline{n-1}\, p^{1-n}x^{n-2}}.$$

1°. Sit $n = 2$: erit $R = -\frac{1}{2}p \times \left(1 + 4\frac{xx}{pp}\right)^{\frac{3}{2}}$; & facta $x = 0$, $R = -\frac{1}{2}p$.

 2°. Sit

2°. Sit $n > 2$: erit $R = \frac{\left(1+nn\frac{x^{2n-2}}{p^{2n-2}}\right)^{\frac{3}{2}}}{-n.\overline{n-1}\frac{x^{n-2}}{p^{n-1}}}$; & facta $x=0$, $R=-\frac{p^{n-1}}{n(n-1)0^{n-2}}$, quod eſt ſignum impoſſibilis. Nullus eſt circulus, utut magno radio deſcribatur, curvam in vertice contingens, qui extra curvam cadat ad partes vertici vicinas.

3°. Sit $n \begin{smallmatrix}< 2\\ > 1\end{smallmatrix}$; erit $R = -p^{n-1}x^{2-n} \times \frac{\left(1+nn\frac{x^{2n-2}}{p^{2n-2}}\right)^{\frac{3}{2}}}{n.\overline{n-1}}$: proinde facta $x=0$, eſt $R=0$; ſeu nullus eſt circulus, radio utut parvo deſcriptus, qui curvam in vertice ita contingat, ut ad partes vertici vicinas intra curvam cadat.

4°. Sit $n=1$: erit $R = \frac{2^{\frac{3}{2}}}{0} \times x$. Hoc caſu linea propoſita eſt recta: proinde contradictorium eſt, de curvatura ejus dicere; quod monemur ſigno impoſſibilitatis $\frac{1}{0}$, ab abſciſſa x independente. Idem contingit, ſi ſit $n=0$; quo caſu linea recta parallela eſt axi, ad quem refertur.

5°. Sit $n \begin{smallmatrix}< 1\\ > 0\end{smallmatrix}$: permutatis coordinatis, caſus hic ad tres priores reducitur.

6°. Sit n negativa, ſeu curva propoſita ſit hyperbolica.

Tunc $R = -\frac{(x^{2+2n}+nnp^{2+2n})^{\frac{3}{2}}}{n.\overline{n+1}p^{1+n}} \times \frac{1}{x^{2n+1}}$. Et quoniam contradictoria eſt ſuppoſitio $x=0$; introductione ſigni $\frac{1}{0^{2n+1}}$ monemur, contradictorium eſſe, loqui de curvatura curvæ in puncto per abſurdum ficto, ubi aſymptoto occurrat.

Quo minor eſt x, eo magis radii curvaturæ valor accedit ad $R=-\frac{nn}{n+1}\frac{p^{2n+2}}{x^{2n+1}}$; & quoniam nullus eſt abſciſſæ x limes quod ad parvitatem, nullus etiam eſt radii R limes quod ad magnitudinem.

Sit $ppy = xx(a-x)$. Erit $pp\frac{dy}{dx} = x(2a-3x)$

$$pp\frac{ddy}{dx^2} = 2(a-3x) \qquad R=\frac{pp}{2(3x-a)}\left(1+\frac{xx(2a-3x)^2}{p^4}\right)^{\frac{3}{2}}$$

Fiat

Fiat $3x = a$: erit $R = \frac{pp}{2a.0}\left(1+\frac{a^4}{9p^4}\right)^{\frac{3}{2}}$. Huic abſciſſæ reſpondet punctum flexus contrarii. (Vid. Fig. 47.)

Sit $p^3y = x^3(a-x)$. Tunc $p^3\frac{dy}{dx} = xx(3a-4x)$

$$p^3\frac{ddy}{dx^2} = 2x(3a-6x) \qquad R = \frac{p^3}{2x(6x-3a)}\left(1+\frac{xx(3a-4x)^2}{p^6}\right)^{\frac{3}{2}}.$$

Utrique abſciſſæ $\begin{matrix} x = 0 \\ x = \frac{1}{2}a \end{matrix}$ reſpondent puncta flexus contrarii, quibus nullus reſpondet limes quoti $\frac{Mt^2}{M't}$. (Vid. Fig. 48.)

In puncto flexus contrarii ob $\frac{ddy}{dx^2} = 0$ eſt $\frac{dy}{dx} = C$, $y = C' + Cx$. Proinde æquatio curvæ eo propius accedit ad æquationem lineæ rectæ, quo puncta curvæ ad punctum flexus contrarii propius accedunt.

§. 196. Conſideratio curvaturæ curvarum ad focum aliquem relatarum in matheſi, mixta præſertim, frequenter magni momenti applicationibus inſervit; quare breviter eam explicabo.

Sit F focus, ad quem curva MM' refertur per radios vectores FM, FM', atque angulos AFM, AFM'. Per M agatur recta tangens MT, cui FM' in t occurrat. Curva eſt verſus focum F $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$, prouti $Ft \gtrless FM'$. Fig. 57.

Sit $FM = y$, $FM' = y'$, $AFM = x$, $MFM' = \Delta x$.

Itaque $y' = y + \frac{\Delta x}{1}\cdot\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2} + \frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3} + \frac{\Delta x^4}{1\ldots4}\cdot\frac{d^4y}{dx^4} + \ldots;$

$$Ft = FM\frac{\text{ſin.}FMT}{\text{ſin.}(FMT-\Delta x)} = y \times \frac{1}{\text{coſ.}\Delta x - \text{cot.}FMT\,\text{ſin.}\Delta x}$$

$$= y\,\text{ſec.}\Delta x(1+\text{cot.}FMT\,\text{tang.}\Delta x + \text{cot.}^2FMT\,\text{tang.}^2\Delta x + \text{cot.}^3FMT\,\text{tang.}^3\Delta x + \ldots)$$

$$= y(1+\tfrac{1}{2}\text{tang.}^2\Delta x - \tfrac{1}{2}\cdot\tfrac{1}{1}\cdot\tfrac{1}{2}\text{tang.}^4\Delta x\ldots)(1+\text{cot.}FMT\,\text{tang.}\Delta x + \text{cot.}^2FMT\,\text{tang.}^2\Delta x + \ldots)$$

$$= y + y\,\text{cot.}FMT\,\text{tang.}\Delta x + \tfrac{1}{2}y\,\text{tang.}^2\Delta x + \tfrac{1}{2}y\,\text{cot.}FMT\,\text{tang.}^3\Delta x + \ldots$$
$$+ y\,\text{cot.}^2FMT\,\text{tang.}^2\Delta x + y\,\text{cot.}^3FMT\,\text{tang.}^3\Delta x \ldots$$

Atqui $\text{cot.}FMT = \frac{1}{y}\frac{dy}{dx}$ (§. 49.). Ergo, curva poſita verſus focum concava, eſt

 $M't$

$$M't = \frac{dy}{dx}(\text{tang.}\Delta x\text{-}1) + \tfrac{1}{2}y\,\text{tang.}^2\Delta x \qquad + \tfrac{1}{2}\frac{dy}{dx}.\,\text{tang.}^3\Delta x + \ldots$$
$$+ \frac{1}{y}.\left(\frac{dy}{dx}\right)^2\text{tang.}^2\Delta x + \frac{1}{yy}.\left(\frac{dy}{dx}\right)^2.\,\text{tang.}^3\Delta x$$
$$- \frac{1}{1.2}.\frac{ddy}{dx^2}.\Delta x^2 \qquad - \frac{1}{1.2.3}.\frac{d^3y}{dx^3}.\Delta x^3$$

Proinde quoti $\frac{M't}{\Delta x^2}$ limes (ſi quis detur) eſt $\frac{1}{2}y + \frac{1}{y}.\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2}$.

Scilicet curva verſus focum eſt $\begin{matrix}\text{concava}\\ \text{convexa}\end{matrix}$, ſi $\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2} \gtrless 0$. Et poſito curvam ad utramque puncti M partem extendi, M eſt punctum flexus contrarii, ſi ſit $\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}.\frac{ddy}{dx^2} = 0$.

Ab radio vectore MF abſcindatur $Mq = \lim.\frac{Mt^2}{M't} = \frac{yy + \left(\frac{dy}{dx}\right)^2}{\frac{1}{2}y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{1.2}\frac{ddy}{dx^2}}$; & ex q puncto erigatur perpendiculum qR, quod normali per M ductæ in R occurrat: erit MR diameter curvaturæ in puncto M (quod demonſtratur, uti §.193.). Sed $MR = Mq\,\text{ſec.}\,qMR = Mq\,\text{coſec.}\,FMT = \frac{Mq}{\text{ſin.}\,FMT} = Mq \times \frac{\sqrt{\left(yy + \left(\frac{dy}{dx}\right)^2\right)}}{y}$. Proinde

$MR = \frac{\left(yy + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{1}{2}yy + \left(\frac{dy}{dx}\right)^2 - \frac{1}{2}y\frac{ddy}{dx^2}}$; & recta MR bifariam diviſa in Z, fit radius curvaturæ $MZ = \frac{\left(yy + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{yy + 2\left(\frac{dy}{dx}\right)^2 - y\frac{ddy}{dx^2}}$.

Formulas has variis exemplis illuſtrabo.

Exemplum primum. Sit $y = r$ quantitas conſtans: tunc $MZ = \frac{y^3}{yy} = y$, quæ eſt æquatio circuli.

Exemplum ſecundum. Sit $y = p\,\text{ſec.}^2\tfrac{1}{2}x$, quæ eſt æquatio focalis parabolæ conicæ. $\frac{dy}{dx} = p\,\text{ſec.}^2\tfrac{1}{2}x\,\text{tang.}\tfrac{1}{2}x = y\,\text{tang.}\tfrac{1}{2}x$

$\frac{ddy}{dx^2} = p(\text{ſec.}^2x\,\text{tang.}^2\tfrac{1}{2}x + \tfrac{1}{2}\text{ſec.}^4\tfrac{1}{2}x)$ $\qquad \lim.\frac{M't}{\Delta x^2} = \frac{1}{4}.\frac{yy}{p}$

Mq

$$Mq = \frac{yy + yy\,\text{tang.}^2\tfrac{1}{2}x}{\tfrac{1}{4}\frac{yy}{p}} = 4p\,\text{ſec.}^2\tfrac{1}{2}x = 4y$$

$$MZ = \frac{y^3\,\text{ſec.}^3\tfrac{1}{2}x}{\tfrac{1}{2}\frac{y^3}{p}} = 2p\,\text{ſec.}^3\tfrac{1}{2}x = 2y\,\text{ſec.}\tfrac{1}{2}x = 2y\sqrt{\frac{y}{p}}.$$

Exemplum tertium. Sit $y = \frac{bb}{a+e\,\text{coſ.}x}$, quæ eſt æquatio ellipſis conicæ ad focum relatæ. Fit $\frac{dy}{dx} = \frac{bbe\,\text{ſin.}x}{(a+e\,\text{coſ.}x)} = e\,\text{ſin.}x \times \frac{yy}{bb}$ $\text{lim.}\frac{M't}{\Delta x^2} = \tfrac{1}{2}a.\frac{yy}{bb}$.

$$\frac{ddy}{dx^2} = \frac{bbe\,\text{coſ.}x}{(a+e\,\text{coſ.}x)^2} + \frac{2bbee\,\text{ſin.}^2x}{(a+e\,\text{coſ.}x)^3}$$

$$Mq = \frac{yy + ee\,\text{ſin.}^2x\,\frac{y^4}{b^4}}{\tfrac{1}{2}a\,\frac{yy}{bb}} = 2\frac{y(2a-y)}{a}.$$

$$MZ = \frac{\sqrt{\left(yy + \left(\frac{dy}{dx}\right)^2\right)}}{y} \times \frac{y(2a-y)}{a} = \sqrt{\left(1 - \frac{y(2a-y)}{bb}\right)} \times \frac{y(2a-y)}{a}.$$

Eædem formulæ ad hyperbolam quoque applicantur.

Exemplum quartum. Sit $y = r^{\frac{x}{\varphi}}$, quæ eſt æquatio ſpiralis logarithmicæ. Fit $\frac{dy}{dx} = \frac{1}{\varphi}y.\frac{1}{\varphi}\log.r$ $\text{lim.}\frac{M't^2}{\Delta x^2} = \tfrac{1}{2}y(1 + \frac{1}{\varphi\varphi}\log.^2r)$

$\frac{ddy}{dx^2} = y.\frac{1}{\varphi\varphi}\log.^2r$ $\text{lim.}\frac{Mt^2}{M't} = 2y$ Proinde limites hi pariter creſcunt in progreſſione geometrica.

$MZ = y\sqrt{(1 + \frac{1}{\varphi\varphi}\log.^2r)}$. Radii igitur curvaturæ etiam in progreſſione geometrica creſcunt.

Exemplum quintum. Sit $y = a\,\text{ſec.}x \pm b$, quæ eſt æquatio conchoidis.

$$\frac{dy}{dx} = a\,\text{ſec.}x\,\text{tang.}x$$

$$\frac{ddy}{dx^2} = a\,\text{ſec.}x\,\text{tang.}^2x + a\,\text{ſec.}^3x = a\,\text{ſec.}x(\text{tang.}^2x + \text{ſec.}^2x)$$

$$MZ = \frac{(aa\,\text{ſec.}^4x \pm 2ab\,\text{ſec.}x + bb)^{\frac{3}{2}}}{\mp 2ab\,\text{ſec.}x\,\text{tang.}^2x \pm ab\,\text{ſec.}x + bb}.$$

1°. Sit

1°. Sit $b=0$: linea, cujus æquatio $y=a\,\mathrm{ſec.}x$, recta eſt; de cujus curvatura dicere contradictorium eſt, quod monet introductio ſigni impoſſibilis $\frac{1}{0}$.

2°. Sit $b=a$: eſt $MZ=a\dfrac{(\mathrm{ſec.}^4x\pm 2\,\mathrm{ſec.}x+1)^{\frac{3}{2}}}{\mp 2\,\mathrm{ſec.}x(\mathrm{tang.}^2x\mp 1)+1}$.

Fiat $x=0$: erit $MZ=\dfrac{\frac{2}{3}a}{\frac{0}{3}a}$. Nempe conchois inferior caſu, quo $b=a$ & $x=a$, in polo cuſpidem habet; & dum ad polum accedit, nullus eſt radii curvaturæ limes quod ad parvitatem. Vid. Fig. 58.

3°. Conchois ſuperior habet punctum flexus contrarii, quod determinatur per æquationem $2a\,\mathrm{ſec.}\,x\,\mathrm{tang.}^2x=a\,\mathrm{ſec.}\,x+b$. Fiat $b=a$: erit $2\,\mathrm{ſec.}x\,\mathrm{tang.}^2x=a(\mathrm{ſec.}x+1)$; unde $2\,\mathrm{ſec.}x(\mathrm{ſec.}x-1)=1$, & $\mathrm{ſec.}x=\begin{cases}\dfrac{\sqrt{3}+1}{2}\\[4pt] -\dfrac{\sqrt{3}-1}{2}\end{cases}$, quorum valorum prior tantum propoſito ſatisfacit.

4°. Facto $x=90°$, fit $MZ=a\times\dfrac{\mathrm{ſec.}^6 90°}{\mathrm{ſec.}90°\,\mathrm{tang.}^2 90°}=a\,\mathrm{ſec.}^3 90°=a\times\infty^3$. Signum hoc impoſſibilis monet: contradictorium eſſe de curva loqui caſu, quo $x=90°$; nempe conchois curva eſt aſymptotica, cujus aſymptota eſt axi perpendicularis.

§. 197. Cum doctrina curvaturæ arctiſſime connectitur, quam primus geometrica methodo tradidit celeb. Hugenius, theoria curvarum evolutione genitarum; ſtrictim itaque hoc loco adhuc exponenda.

Fig. 59. Curvæ AMM' filum ſeu linea flexilis circumplicata intelligatur; &, manente una extremitate illi affixa, altera extremitas, ſtylo ex. gr. illi annexo, ſenſim ita abduci concipiatur, ut pars fili MN, quæ ſoluta eſt, ſemper in directum extenſa ſit, & curvam tangat in puncto M, ubi illam deſerit. Curva ANN', quam ſtylus motu hoc deſcribit, dicitur *evolutione* curvæ AMM' *genita*; ipſa vero AMM' dicitur *evoluta*.

Obſervatio. Filum curvæ AMM' applicatum poteſt in A terminari; quo caſu curva evolutione genita tranſit per punctum A. Sed fieri etiam poteſt, ut filum ultra curvæ punctum A protendatur juxta rectam AA' poſitione & magnitudine datam, quæ ſcilicet curvam AMM' in puncto A contingit. Quo caſu curva evolutione genita tranſit per punctum A'.

Centro M radio MN deſcribatur circulus, & agatur recta NT ipſum in N tangens. Dico: rectam NT in eodem puncto N contingere curvam evolutione deſcriptam.

1°. Sit N' punctum curvæ NN' remotius a principio A, quam eſt punctum N. Sit $N'M'$ ſitus fili, quo extremum ejus pervenit in N'; & ſit M' punctum curvæ evolutæ AMM', ubi filum $N'M'$ eam contingit: rectæ NM, $N'M'$ ſibi invicem occurrant in puncto m', & recta $M'N'$ occurrat in n' tangenti NT circuli. In triangulo mixtilineo $MM'm'$ eſt $Mm' + m'M' > MM'$; proinde addito utrinque arcu $MA = MN$, fit $Mm' + m'M' + MN > M'N'$: Fig. 59. 1°.

unde $m'N \quad > m'N'$. Atqui, propter angulum rectum N, $m'n' > m'N$: ergo (a fortiori) $m'n' > m'N'$; proinde punctum n' tangentis NT eſt extra curvam NN'.

2°. Sit N' punctum inter puncta A & N. Sit $N'M'$ ſitus fili, quo extremum ejus eſt in N'. Agatur recta MN', quæ tangenti NT in n' puncto occurrat. In triangulo mixtilineo $MM'N'$ eſt $MM' + M'N' > MN'$, hoc eſt $MN > MN'$; atqui propter angulum rectum N eſt $Mn' > MN$: ergo a fortiori $Mn' > MN'$; ideoque punctum n' eſt extra curvam NN'. Fig. 59. 2°.

Circulus igitur centro M radio MN deſcriptus & curva evolutione genita NN' habent in N puncto tangentem communem; proindeque circulus hic & curva NN' ſeſe invicem contingunt in N.

Porro autem circulus hic & curva NN' ſeſe invicem in N puncto ita contingunt, ut prior poſteriorem oſculetur, ſeu ut nullus alius circulus, curvam in N contingens, inter hanc curvam & priorem circulum transire poſſit. Etenim

1°. Sit M' punctum quodvis curvæ AMM' ultra punctum M ſitum; ac ſit $M'N'$ ſitus fili, quo curvam MM' in M' tangit, eodemque ſitu filum in puncto R' occurrat circumferentiæ radio MN deſcriptæ. In triangulo mixtilineo $M'MR$ eſt $M'M + MR' > M'R'$ Fig. 59. 1°.

ſeu $M'N' \quad > M'R'$; proinde circumferentia centro M radio MN deſcripta cadit intra curvam NN'; & nulla circumferentia, radio minore quam MN deſcripta, atque curvam in N contingens, cadit inter arcum curvæ NN'

& arcum circularem NR'. Sit autem Nm' radius major quam NM; atque filum per m' tranſiens curvis MM', NN', NR' in M', N', R' punctis reſpective occurrat. Quoniam $Mm'+m'M' > MM'$

eſt $Nm'+m'M' > M'N'$:

unde $m'N > m'N'$; ideoque circulus centro m', radio $m'N$ deſcriptus extra curvam NN' cadit.

Fig. 59. 2°. 2°. Sit M' punctum quodvis curvæ AM inter puncta A & M ſitum; ſit rurſus $M'N'$ ſitus fili, quo curvam MM' tangit; & hoc ſitu filum occurrat in R' circumferentiæ NR', & curvæ NN' in N' puncto.

In triangulo mixtilineo $MM'R'$ eſt $MM'+M'R' > MR' > MN > MM'+M'N'$:

unde $M'R' > M'N'$;

proinde circumferentia NR' cadit extra curvam NN': & a fortiori nulla circumferentia curvam NN' contingens in N, & radio majore quam MN deſcripta, inter arcus NN', NR' tranſire poteſt. Sit autem Nm' radius minor quam NM; & ſit $m'M'N'$ ſitus fili, quo per m' punctum tranſit. Quoniam

$Mm'+m'M' > MM'$; addito utrinque arcu $M'A$, eſt

$Mm'+m'N' > MM'N' > MN$:

unde $m'N' > m'N$. Proinde circulus centro m' radio $m'N$ deſcriptus cadit intra curvam NN'.

Circumferentia igitur circuli, centro M, radio MN deſcripti, curvam oſculatur in N puncto.

§. 198. Curva itaque AMM' locus eſt centrorum circulorum curvam ANN' oſculantium. Si curva NN' in aliquo puncto nullum habet radium curvaturæ, ideo quod nullus ſit radii hujus limes quod ad parvitatem; curvæ MM', NN' hoc puncto ſibi invicem occurrunt: ſi vero curva NN' in aliquo puncto nullum habet radium curvaturæ, ideo quod nullus ſit radii hujus limes quod ad magnitudinem; nullus etiam eſt limes diſtantiæ, qua curva MM' ab curva NN' removetur. Quando autem aliquis eſt radii curvaturæ limes quod ad parvitatem, curva MM' curvæ NN' non occurrit; & curva MM' intra limites quosdam continetur, ſi ſimul aliquis eſt radii curvaturæ curvæ NN' limes quod ad magnitudinem.

Sit

Sit Z centrum circuli ejusdem cum curva SM curvaturæ in puncto M; ex quo demittatur in axem SP recta perpendicularis ZX. Data æquatione curvæ SM, determinatur etiam æquatio curvæ, quæ eſt locus punctorum Z, per coordinatas SX, ZX. Etenim ſit ZQ ipſi MP perpendicularis. Erunt Fig. 56.

$$ZX = MQ - PQ = \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}} - y$$

$$SX = SP + ZQ = SP + MQ \,\text{tang.}\, ZMQ = x + \frac{dy}{dx} \times \frac{1+\left(\frac{dy}{dx}\right)^2}{-\frac{ddy}{dx^2}}.$$

Exemplum primum. Sit $yy = rr - xx$: $\frac{dy}{dx} = -\frac{x}{y}$;

$$\frac{ddy}{dx^2} = -\frac{1}{y} + \frac{x\frac{dy}{dx}}{yy} = -\frac{1}{y} - \frac{xx}{y^3} = -\frac{rr}{y^3};$$

$$ZX = \frac{1+\frac{xx}{yy}}{\frac{rr}{y^3}} - y = y - y = 0;$$

$SX = x - \frac{x}{y} \times x = x - x = 0$. Proinde puncta Z ad punctum unicum reducuntur; quod conſentit cum palmaria circumferentiæ circuli proprietate.

Exemplum ſecundum. Sit $y = \sqrt{(2rx - xx)} + r\,\text{arc.ſin.v.}\frac{x}{r}$:

$$\text{erit } \frac{dy}{dx} = \sqrt{\frac{2r-x}{x}}$$

$$\frac{ddy}{dx^2} = -\frac{r}{x} \times \frac{1}{\sqrt{(2rx-xx)}}$$

$$ZX = -\left(r\,\text{arc.ſin.v.}\frac{x}{r} - \sqrt{(2rx-xx)}\right)$$

$$SX = 4r - x.$$

Unde facile deducitur: cycloidis vulgaris evolutam pariter eſſe cycloidem vulgarem, eamque cum priore congruentem.

Exemplum tertium. Sit $yy = 2px$: $\frac{dy}{dx} = \frac{p}{y}$; $\frac{ddy}{dx^2} = -\frac{p\frac{dy}{dx}}{yy} = -\frac{pp}{y^3}$.

 hinc

Hinc $ZX = \frac{y^3}{pp}$, $SX = p + 3x$.

$$ZX^2 = \frac{y^6}{p^4} = \frac{8p^3x^3}{p^4} = 8\frac{x^3}{p} = 8\frac{\left(\frac{SX-p}{3}\right)^3}{p}\text{ ; } p.ZX^2 = \tfrac{8}{27}(SX-p)^3.$$ Parabolæ igitur conicæ evoluta & ipſa parabola eſt æquatione $pyy = x^3$ determinata.

Contra determinatio curvæ evolutione genitæ requirit rectificationem curvæ evolutæ. Et cum curvæ propoſitæ unica reſpondeat evoluta: uni eidemque curvæ reſpondent innumeræ curvæ evolutione hujus genitæ, quatenus variatur fili evolventis principium.

§. 199. Quamdiu curva evoluta non habet punctum flexus contrarii, curva evolutione ejus genita continue verſus easdem partes progreditur. Quodſi autem curva evoluta punctum aliquod flexus contrarii habet; dum filum abducitur a punctis curvæ ſenſu oppoſito flexæ, ſtylus ad partes oppoſitas regreditur: unde curva evolutione genita duobus conſtat ramis, in puncto regreſſus in cuſpidem coëuntibus, quorum uterque verſus easdem partes $\begin{matrix}\text{concavus}\\ \text{convexus}\end{matrix}$ eſt. Quare doctrina evolutionis curvarum apta eſt ad dirimendam controverſiam de hoc curvarum genere inter mathematicos agitatam, & ab ill. Eulero in *Commentariis Acad. Berol. ad ann.* 1749. ſolide enodatam. Hic breviſſime argumentum illud attingere ſufficiet.

Sit n numerus fractus ſpurius poſitivus, cujus denominator eſt numerus par. Sit $py = \varphi'x \pm \varphi x . x^n$; ita ut functiones $\varphi'x$, φx factorem x non comprehendant. Per hypotheſin x non poteſt eſſe negativa.

Sint

$$\varphi'x = A' + B'x + C'x^2 + D'x^3 + \dots$$
$$\varphi x = A + Bx + Cx^2 + Dx^3 + \dots$$

Igitur

$$py = A' + B'x + C'x^2 + D'x^3 + \dots \pm (Ax^n + Bx^{n+1} + Cx^{n+2} + Dx^{n+3} + \dots)$$
$$p\frac{dy}{dx} = B' + 2C'x + 3D'x^2 + \dots \pm (nAx^{n-1} + (n+1)Bx^n + (n+2)Cx^{n+1} + (n+3)Dx^{n+2} + \dots)$$
$$p\frac{ddy}{dx^2} = 2C' + 3.2D'x + \dots \pm (n.n-1\,Ax^{n-2} + n+1.n\,Bx^{n-1} + n+2.n+1\,Cx^n + n+3.n+2\,Dx^{n+1} + \dots)$$

Poſito

Poſito $n > 2$; ramus uterque $\genfrac{}{}{0pt}{}{\text{concavus}}{\text{convexus}}$ eſt verſus axem ad partes abſciſſæ $x = 0$ proximas, prouti C' eſt numerus $\genfrac{}{}{0pt}{}{\text{negativus}}{\text{poſitivus}}$. Et ordinata abſciſſæ $x = 0$ reſpondens $\genfrac{}{}{0pt}{}{\text{major}}{\text{minor}}$ eſt ordinatis ipſi proximis utroque ramo terminatis, prouti B' numerus eſt $\genfrac{}{}{0pt}{}{\text{negativus}}{\text{poſitivus}}$ (poſito A' poſitivo).

Caput vicesimum.

De problematibus, quæ vocantur, isoperimetricis.

§. 200.

Problemata, quibus caput hoc deſtinatur, præcipuos tam vergentis ſuperioris, quam præſentis ſeculi mathematicos occuparunt; & nonnulla eorum materiam ipſis ſuggeſſere provocationum, quibus vires ſuas mutuo tentarunt. Poſt Joh. Bernoullium (*a*) cel. Fontaine jam inde ab anno 1732. ad methodum aliquam generalem problemata hæc ſolvendi eniti allaboravit. (*b*). Ill. Eulerus eximium de illis anno 1744. edidit tractatum, inſcriptum: *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.* Sagaciſſimus de la Grange ſolutionem ipſorum ad methodum univerſalem & mere analyticam reduxit, quæ *methodus variationum* nominari conſuevit. (*c*) Quamvis autem diverſi poſt eum ſcriptores methodum hanc tradiderint vel tranſcripſerint; difficile tamen etiamnum mihi videtur principia ejus ita explanare, ut rigori mathematico aſſueti tirones nullibi hæreant.

Cum tamen argumentum hoc inter mathematicos celebre magni ſit in matheſi, inprimis mixta, momenti; præcipuam ſaltem ejus partem diſtincte & accurate exponere, atque ad inſtar ceterarum calculi differentialis & integralis applicationum captui tironum accommodare conatus ſum. Nec diffiteor, me, pro hujus ſcopi ratione, argumentum illud eadem univerſalitate amplexum non

(*a*) Vid. *Memoires de Paris* pour 1706.

(*b*) Vid. *Memoires de Paris* pour 1734. & *Memoires preſentés à l'Academie de Paris* par Mr. Fontaine. Paris 1764.

(*c*) *Memoires de Turin.* T. II.

eſſe, qua ab EULERO & DE LA GRANGE fuit pertractatum. Veruntamen ſatis me oſtendiſſe ſpero, fieri poſſe, ut ad ſimplicia & ſolida methodi limitum principia reducatur.

Expoſitionem meam ita inſtituam, ut ſingularia primum evolvam problemata; tum a caſibus particularibus ad generaliores paulatim annitar.

Fig. 60. §. 201. *Problema.* Sint A & A' duo puncta poſitione data; ſit etiam PM recta poſitione data; ac ſint F, F' duo factores magnitudine dati: quæritur punctum M tale, ut $F \times AM + F' \times MA'$ ſit omnium minimum.

A punctis A, A' in rectam MP demittantur perpendicula Aa, $A'a'$; & ſint $Aa = b$, $A'a' = b'$, $aa' = a$. Sint etiam $AM = z$, $A'M = z'$, $aM = y$, $a'M = y'$.

Quoniam $Fz + F'z' =$ minimo;

oportet, ſit $F\dfrac{dz}{dv} + F'\dfrac{dz'}{dv} = 0$.

Atqui $zz = bb + yy$ ergo $\dfrac{dz}{dv} = \dfrac{y}{z}\cdot\dfrac{dy}{dv} = \dfrac{dy}{dv}\text{coſ.}AMa$

$z'z' = b'b' + y'y'$ $\dfrac{dz'}{dv} = \dfrac{y'}{z'}\cdot\dfrac{dy'}{dv} = \dfrac{dy'}{dv}\text{coſ.}A'Ma'$.

Quare $F\dfrac{dy}{dv}\text{coſ.}AMa + F'\dfrac{dy'}{dv}\text{coſ.}A'Ma' = 0$.

Atqui $y + y' (= a)$ datur magnitudine. Igitur $\dfrac{dy}{dv} + \dfrac{dy'}{dv} = 0$;
& hinc $F\text{coſ.}AMa = F'\text{coſ.}A'Ma'$.

Proinde ſumma $Fz + F'z'$ eſt omnium minima, quando $F\text{coſ.}AMa = F'\text{coſ.}A'Ma'$.

Exemplum. Sit $F = F'$: caſu igitur, quo $z + z'$ eſt omnium minima, fit coſ.AMa = coſ.$A'Ma'$; proinde $AMa = A'Ma'$; & puncta A, M, A' jacent in directum.

Corollarium. Per puncta data A, A' agantur duæ rectæ AB, $A'B'$, poſitione datæ MP parallelæ: erit $F\text{coſ.}MAB = F'\text{coſ.}A'MP$, caſu, quo ſumma $Fz + F'z'$ eſt omnium minima.

Fig. 61. §. 202. Ducatur quæcunque recta, quæ (v. gr.) ſit rectis AB, $A'B'$ perpendicularis, atque ipſis in punctis B, B' occurrat. Dividatur BB' in partes quot-

quotcunque æquales in punctis P, P', P'', P''', P^{iv}.... Per puncta diviſionum agantur totidem rectæ ipſi BB' perpendiculares. Sint F, F', F'', F''', F^{iv}.... coefficientes dati; & determinanda ſint in perpendiculis iſtis puncta M, M', M'', M''', M^{iv}.... ita poſita, ut ductis rectis

AM, MM', $M'M''$, $M''M'''$, $M'''M^{iv}$.... quæ dicantur reſpective

z, z', z'', z''', z^{iv} .... ſumma

$Fz + F'z' + F''z'' + F'''z''' + F^{v}z^{iv} + \ldots$ ſit omnium minima.

Ut huic conditioni ſatisfiat, debet eſſe

$$\begin{aligned} F\,\text{coſ.}MAB &= F'\,\text{coſ.}M'MP \\ &= F''\,\text{coſ.}M''M'P' \\ &= F'''\,\text{coſ.}M'''M''P'' \\ &= F^{v}\,\text{coſ.}M^{iv}M'''P''' \\ &= - \; - \; - \; - \end{aligned}$$

Pro vertice igitur quocunque M, inter duos vertices $'M$, M' poſito, erit $F\,\text{coſ.}M'MP$ quantitas conſtans.

Sit S punctum aliquod in axe BB' (producto v. gr.); & ſint F, F', F'', F''', F^{iv}.... functiones quædam ſimiles abſciſſarum SP, SP', SP'', SP''', SP^{iv}.... Laterum figuræ, ſeu axis BB' partium, numero manente eodem; pariter eſt $F\,\text{coſ.}M'MP$ quantitas conſtans.

Proinde, aucto partium axis numero, in curva etiam, quæ limes eſt polygonorum $AMM'M''M'''M^{iv}$.... quantitatis $F\,\text{coſ.}M'MP$ limes erit quantitas aliqua conſtans, quæ dicatur C. Itaque $F\frac{dy}{dz} = C$. Hinc $\frac{dz}{dy} = \frac{F}{C}$;

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{FF}{CC};\ \left(\frac{dx}{dy}\right)^2 = \frac{FF-CC}{CC};\ \frac{dy}{dx} = \frac{C}{\sqrt{(FF-CC)}}.$$

Exiſtente igitur X functione quadam abſciſſarum x, & deſignante z arcum hujus curvæ: ſi fuerit $\frac{dZ}{dz} = X$; & quæratur curva, in qua Zz ſit omnium minima: eſt $\frac{dy}{dz} = \text{coſ.}TMP = \frac{C}{X}$; unde $\frac{dy}{dx} = \frac{C}{\sqrt{(XX-CC)}}$.

Exemplum primum. Sit $X = \sqrt{x}$: erit $\frac{dy}{dx} = \frac{\sqrt{c}}{\sqrt{(x-c)}}$; $y = c' + \sqrt{(c(x-c))}$. Sit $y = 0$, quando $x = c$: erit $c' = 0$, $y = \sqrt{(c(x-c))}$. Æquatio hæc eſt ad parabo-

rabolam; & minimum hoc pertinet ad motum projectilium terreſtrium in ſpatio libero.

Exemplum ſecundum. Sit $X = \frac{1}{\sqrt{x}}$: fit $-\frac{dy}{dx} = \sqrt{\frac{x}{c-x}}$, $y = c' + \sqrt{(cx - xx)} - \frac{1}{2}c \text{ arc.ſin.v.} \frac{x}{\frac{1}{2}c}$; æquationes cycloidis vulgaris. Sit $x = 0$: tum $\frac{dy}{dx} = 0$; ſeu tangens curvæ ſit axi parallela.

Sit $y = 0$, quando $x = c$: tum $c' = 0$, & $\frac{dy}{dx} = \frac{1}{0}$; ſeu tangens fit axi perpendicularis. Minimum hoc refertur ad deſcenſum omnium celerrimum.

§. 203. In §. 201. loco ſummæ $Fz + F'z'$ proponatur ſumma $Fz^n + F'z'^n$, quæ debeat eſſe omnium minima.

Erit eodem modo $nFz^{n-1}\frac{dz}{dv} + nF'z'^{n-1}\frac{dz'}{dv} = 0$

$$\text{ſeu} \quad Fz^{n-1}\frac{dz}{dv} + F'z'^{n-1}\frac{dz'}{dv} = 0:$$

$$\text{unde} \quad Fz^{n-1}\text{coſ.}MAB = F'z'^{n-1}\text{coſ.}A'MP.$$

Fiat eadem conſtructio, quæ §. 202. In polygono $AMM'M''M'''M^{\text{iv}} \ldots$, in quo ſumma $Fz^n + F'z'^n + F''z''^n + F'''z'''^n + \ldots$ eſt omnium minima, eſt

$$\begin{aligned} Fz^{n-1}\text{coſ.}MAB &= F'z'^{n-1}\text{coſ.}M'MP \\ &= F''z''^{n-1}\text{coſ.}M''M'P' \\ &= F'''z'''^{n-1}\text{coſ.}M'''M''P'' \\ &= F^{\text{iv}}z^{\text{iv}\,n-1}\text{coſ.}M^{\text{iv}}M'''P''' \\ &= \text{- - - - -} \end{aligned}$$

Quare pro vertice quolibet M, inter duos vertices $'M$, M' poſito, eſt $Fz^{n-1}\text{coſ.}M'MP$ quantitas conſtans. Sit pars quælibet axis $PP' = \Delta x$: erit etiam $F\frac{z^{n-1}}{\Delta x^{n-1}}\text{coſ.}M'MP$ quantitas conſtans. Unde, aucto partium numero, in curva, quæ limes eſt figurarum rectilinearum, quantitatis $F\frac{z^{n-1}}{\Delta x^{n-1}}\text{coſ.}M'MP$ limes erit quantitas conſtans; ſeu $F\left(\frac{dz}{dx}\right)^{n-1}\frac{dy}{dz}$ eſt quantitas conſtans.

Exemplum primum. Sit F quantitas conſtans, & $n = 2$: erit $\frac{dz}{dx}\cdot\frac{dy}{dz} = c$; proinde $\frac{dy}{dx} = c$; cui æquationi ſatisfacit linea recta. Generatim, poſito F conſtante, eſt $\left(\frac{dz}{dx}\right)^{n-1}\cdot\frac{dy}{dz} = c$; æquatio ad lineam rectam.

Exemplum ſecundum. Sit F ipſi x proportionalis. Erit $x\left(\frac{dz}{dx}\right)^{n-1}\cdot\frac{dy}{dz}=c$.

Caſu particulari, quo $n=-2$, fit $x\left(\frac{dx}{dz}\right)^{3}\cdot\frac{dy}{dz}=c$; quæ eſt æquatio curvæ genitricis ſolidi, quod reſiſtentiam omnium minimam patitur a fluido, in quo juxta directionem axi rotationis parallelam movetur.

§. 204. Loco ſummæ $Fz^{n}+F'z'^{n}$ proponatur ſumma $F\varphi z+F'\varphi z'$, quæ omnium minima eſſe debeat: poſitis φz, $\varphi z'$ ſimilibus ipſarum z, z' functionibus, talibus, ut $\frac{d\varphi z}{dv}=\pi z\frac{dz}{dv}$, ideoque $\frac{d\varphi z'}{dv}=\pi z'\frac{dz'}{dv}$; in quibus πz & $\pi z'$ ſunt etiam functiones ipſarum z & z' ſimiles.

Erit eodem modo $F\pi z\frac{dz}{dv}+F'\pi z'\frac{dz'}{dv}=0$.

Igitur $F\pi z\,\text{coſ.}MAB=F'\pi z'\,\text{coſ.}A'MP$. Unde eadem facta conſtructione, quæ §. 202. deducitur, quantitatem $F\pi z\,\text{coſ.}M'MP$ eſſe conſtantem; proinde in curva, quæ limes eſt figuræ rectilineæ $AMM'M''M'''M^{\text{iv}}\ldots$, eſt $\lim.F\pi\frac{\Delta z}{\Delta x}\text{coſ.}M'MP$, ſeu $F.\pi\left(\frac{dz}{dx}\right)\frac{dy}{dz}$ quantitas conſtans. Sit nempe $\frac{dZ}{dx}=X\varphi\frac{dz}{dx}$: erit Z omnium minimum, quando $X\pi\left(\frac{dz}{dx}\right).\frac{dy}{dz}$ eſt quantitas conſtans.

§. 205. Inveſtigatio curvæ maximi minimive proprietate præditæ paulo aliter inſtitui potuiſſet modo ſequente; illi admodum analogo, qui §. 185. 186. traditur.

Sit BB' axis magnitudine datus, in partes quotcunque æquales in P, P', P'', P''', $P^{\text{iv}}\ldots$ punctis diviſus.

Agantur perpendicula	BA,	MP,	$M'P'$,	$M''P''$,	$M'''P'''$,	$M^{\text{iv}}P^{\text{iv}}\ldots$
quæ dicantur	a,	y,	y',	y'',	y''',	$y^{\text{iv}}\ldots$
& rectæ		AM,	MM',	$M'M''$,	$M''M'''$,	$M'''M^{\text{iv}}\ldots$
quæ dicantur		z,	z',	z'',	z''',	$z^{\text{iv}}\ldots$
Quæritur ſumma		$F\varphi z+$	$F'\varphi z'+$	$F''\varphi z''+$	$F'''\varphi z'''+$	$F^{\text{iv}}\varphi z^{\text{iv}}+\ldots$
quæ ſit omnium minima; poſitis		φz,	$\varphi z'$,	$\varphi z''$,	$\varphi z'''$,	$\varphi z^{\text{iv}}\ldots$
functionibus ſimilibus ipſius z talibus, ut exponentes differentiales		$\frac{d\varphi z}{dv}$,	$\frac{d\varphi z'}{dv}$,	$\frac{d\varphi z''}{dv}$,	$\frac{d\varphi z'''}{dv}$,	$\frac{d\varphi z^{\text{iv}}}{dv}\ldots$
ſint reſpective		$\pi z\frac{dz}{dv}$,	$\pi z'\frac{dz'}{dv}$,	$\pi z''\frac{dz''}{dv}$,	$\pi z'''\frac{dz'''}{dv}$,	$\pi z^{\text{iv}}\frac{dz^{\text{iv}}}{dv}\ldots$

 Erit

Erit ideo $F\pi z\frac{\mathrm{d}z}{\mathrm{d}v}+F'\pi z'\frac{\mathrm{d}z'}{\mathrm{d}v}+F''\pi z''\frac{\mathrm{d}z''}{\mathrm{d}v}+F'''\pi z'''\frac{\mathrm{d}z'''}{\mathrm{d}v}+F^{\mathrm{iv}}\pi z^{\mathrm{iv}}\frac{\mathrm{d}z^{\mathrm{iv}}}{\mathrm{d}v}+\ldots=0.$

$$
\begin{aligned}
\text{Hinc}\quad & \frac{\mathrm{d}y}{\mathrm{d}v}(F'\pi z'\,\mathrm{cof.}\,M'MP-F\pi z\,\mathrm{cof.}\,MAB)\\
+ & \frac{\mathrm{d}y'}{\mathrm{d}v}(F''\pi z''\,\mathrm{cof.}\,M''M'P'-F'\pi z'\,\mathrm{cof.}\,M'MP)\\
+ & \frac{\mathrm{d}y''}{\mathrm{d}v}(F'''\pi z'''\,\mathrm{cof.}\,M'''M''P''-F''\pi z''\,\mathrm{cof.}\,M''M'P')\\
+ & \frac{\mathrm{d}y'''}{\mathrm{d}v}(F^{\mathrm{iv}}\pi z^{\mathrm{iv}}\,\mathrm{cof.}\,M^{\mathrm{iv}}M'''P'''-F'''\pi z'''\,\mathrm{cof.}\,M'''M''P'')\\
+ & \;-\;-\;-\;-\;-\;-\;-\;-\\
+ & \;-\;-\;-\;-\;-\;-\;-\;-
\end{aligned}
\quad = 0.
$$

Omnes quantitates mutabiles y, y', y'', y''', $y^{\mathrm{iv}}\ldots$ fiant ſimul conſtantes, una excepta; ac proinde omnes exponentes differentiales $\frac{\mathrm{d}y}{\mathrm{d}v}$, $\frac{\mathrm{d}y'}{\mathrm{d}v}$, $\frac{\mathrm{d}y''}{\mathrm{d}v}$, $\frac{\mathrm{d}y'''}{\mathrm{d}v}\ldots$ ſimul evaneſcant, uno excepto. Erit, ut prius,

$$
\begin{aligned}
F\pi z\,\mathrm{cof.}\,MAB &= F'\pi z'\,\mathrm{cof.}\,M'MP\\
&= F''\pi z''\,\mathrm{cof.}\,M''M'P'\\
&= F'''\pi z'''\,\mathrm{cof.}\,M'''M''P''\\
&= F^{\mathrm{iv}}\pi z^{\mathrm{iv}}\,\mathrm{cof.}\,M^{\mathrm{iv}}M'''P'''\\
&= -\;-\;-\;-\\
&= -\;-\;-\;-
\end{aligned}
$$

Unde eædem, quæ prius, fluunt conſequentiæ.

Occurrunt quandoque caſus, quibus ſatius eſſe videtur inveſtigationem propoſitam poſteriori modo aggredi; uti uno alterove exemplo oſtendam.

§. 206. *Problema.* Omnibus uti in §. 202. poſitis, quæritur figura $BAMM'M''\ldots A'B'$, cujus area per perimetrum $AMM'M''M'''\ldots A'$ diviſa præbeat quotum omnium maximum.

Pars quælibet axis PP' dicatur b.

Erit ideo $\frac{b(a+2y+2y'+2y''+2y'''+2y^{\mathrm{iv}}+\ldots+a')}{z+z+z'+z''+z'''+z^{\mathrm{iv}}+\ldots}=$ maximo.

Hinc

Hinc $2(z+z'+z''+z'''+z^{iv}+\dots)\left(\frac{dy}{dv}+\frac{dy'}{dv}+\frac{dy''}{dv}+\frac{dy'''}{dv}+\frac{dy^{iv}}{dv}+\dots\right)$

$$=(a+2y+2y'+2y''+2y'''+2y^{iv}+\dots)\left(\frac{dz}{dv}+\frac{dz'}{dv}+\frac{dz''}{dv}+\frac{dz'''}{dv}+\frac{dz^{iv}}{dv}+\dots\right)$$

$$\begin{aligned}=(a+2y+2y'+2y''+2y'''+2y^{iv}+\dots)\Big\{&\frac{dy}{dv}(\text{coſ.}M'MP-\text{coſ.}MAB)\\ &+\frac{dy'}{dv}(\text{coſ.}M''M'P'-\text{coſ.}M'MP)\\ &+\frac{dy''}{dv}(\text{coſ.}M'''M''P''-\text{coſ.}M''M'P')\\ &+\frac{dy'''}{dv}(\text{coſ.}M^{iv}M'''P'''-\text{coſ.}M'''M''P'')\\ &+\frac{dy^{iv}}{dv}(\text{coſ.}M^{v}M^{iv}P^{iv}-\text{coſ.}M^{iv}M'''P''')\\ &+\ -\ -\ -\ -\ -\\ &+\ -\ -\ -\ -\ -\Big\}\end{aligned}$$

Omnes quantitates mutabiles fiant ſimul conſtantes, una excepta: erit

$$\begin{aligned}\frac{z+z'+z''+z'''+z^{iv}+\dots}{a+2y+2y'+2y''+2y'''+2y^{iv}+\dots}&=\text{coſ.}M'MP-\text{coſ.}MAB\\ &=\text{coſ.}M''M'P'-\text{coſ.}M'MP\\ &=\text{coſ.}M'''M''P''-\text{coſ.}M''M'P'\\ &=\text{coſ.}M^{iv}M'''P'''-\text{coſ.}M'''M''P''\\ &=\ -\ -\ -\ -\ -\\ &=\ -\ -\ -\ -\ -\end{aligned}$$

Quare pro vertice quolibet M duobus verticibus ${}'M$, M' interjacente eſt

$$\text{coſ.}M'MP-\text{coſ.}M'M'P=\frac{z+z'+z''+z'''+z^{iv}+\dots}{a+2y+2y'+2y''+2y'''+2y^{iv}+\dots}$$

Æquatio hæc locum habet, quicunque ſit numerus laterum figuræ. Axis igitur BB' dividatur in partes quotcunque æquales, quarum quælibet ſit Δx: erit etiam $\frac{\text{coſ.}M'MP-\text{coſ.}M'M'P}{\Delta x}=\frac{z+z'+z''+z'''+z^{iv}+\dots}{\Delta x(a+2y+2y'+2y''+2y'''+2y^{iv}+\dots)}$. Quare & prioris quantitatis limes æqualis eſt limiti poſterioris: unde in curva, propoſita

 maxi-

maximi proprietate prædita, eſt (§. 192. Obſ. 3.) $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \frac{P}{S}$ (denotantibus P & S perimetrum & arcum figuræ). Proinde in curva propoſita radius curvaturæ eſt conſtans; ideoque hæc curva eſt circumferentia circuli.

Alterum exemplum. Sit S punctum aliquod in A axe BB' producto, per quod ducatur recta ipſi BB' perpendicularis; requiritur, ut centrum gravitatis perimetri $AMM'M''M'''M^{iv}$.... ſit reſpectu hujus perpendicularis omnium maxime depreſſum.

Diſtantiæ SB, SP, SP', SP'', SP''', SP^{iv}, ... vocentur reſpective
b, c, c', c'', c''', c^{iv}, ...

Erit ideo $\frac{z(b+c)+z'(c+c')+z''(c'+c'')+z'''(c''+c''')+z^{iv}(c'''+c^{iv})+\dots}{z+z'+z''+z'''+z^{iv}+\dots}$ = maximo.

Compendii cauſa numerator & denominator hujus fractionis vocentur M & P reſpective.

Erit $P\left((b+c)\frac{dz}{dv}+(c+c')\frac{dz'}{dv}+(c'+c'')\frac{dz''}{dv}+(c''+c''')\frac{dz'''}{dv}+(c'''+c^{iv})\frac{dz^{iv}}{dv}+\dots\right)$

$= M\left(\frac{dz}{dv}+\frac{dz'}{dv}+\frac{dz''}{dv}+\frac{dz'''}{dv}+\frac{dz^{iv}}{dv}+\dots\right)$

$$\begin{aligned}
&\text{ſeu } P\Big(\tfrac{dy}{dv}((b+c)\text{coſ.}PMA-(c+c')\text{coſ.}P'M'M) &&M\Big(\tfrac{dy}{dv}(\text{coſ.}PMA-\text{coſ.}P'M'M)\\
&+\tfrac{dy'}{dv}((c+c')\text{coſ.}P'M'M-(c'+c'')\text{coſ.}P''M''M') &&+\tfrac{dy'}{dv}(\text{coſ.}P'M'M-\text{coſ.}P''M''M')\\
&+\tfrac{dy''}{dv}((c'+c'')\text{coſ.}P''M''M'-(c''+c''')\text{coſ.}P'''M'''M'') \quad = &&+\tfrac{dy''}{dv}(\text{coſ.}P''M''M'-\text{coſ.}P'''M'''M'')\\
&+\tfrac{dy'''}{dv}((c''+c''')\text{coſ.}P'''M'''M''-(c'''+c^{iv})\text{coſ.}P^{iv}M^{iv}M''') &&+\tfrac{dy'''}{dv}(\text{coſ.}P'''M'''M''-\text{coſ.}P^{iv}M^{iv}M''')\\
&+\;-\;-\;-\;-\;-\;-\;- &&+\;-\;-\;-\;-\;-\\
&+\;-\;-\;-\;-\;-\;-\;- &&+\;-\;-\;-\;-\;-
\end{aligned}$$

Omnes quantitates mutabiles y, y', y'', y''', y^{iv}... fiant ſimul conſtantes præter unam: erit pro vertice quolibet M, cui reſpondet abſciſſa x, & qui vertices inter M, M' jacet,

P((2x

$$P((2x-\Delta x)\text{coſ.}PM'M-(2x+\Delta x)\text{coſ.}P'M'M) = M(\text{coſ.}PM'M-\text{coſ.}P'M'M);$$

ideoque

$$P(2x\frac{\text{coſ.}PM'M-\text{coſ.}P'M'M}{\Delta x}-(\text{coſ.}PM'M+\text{coſ.}P'M'M))=M\frac{\text{coſ.}PM'M-\text{coſ.}P'M'M}{\Delta x};$$

unde & limites ſunt inter ſe æquales, nempe

$$2P\left(x\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}+\frac{dy}{dz}\right) = M\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}$$

Igitur $2Px\frac{dy}{dz} = M\left(C+\frac{dy}{dz}\right)$

$$\frac{dy}{dz} = \frac{MC}{2Px-M}$$

unde $\frac{dy}{dz} = \frac{MC}{\sqrt{((2Px-M)^2-MMCC)}}$.

Obſervatio. In duobus his exemplis maximi minimive proprietas intra limites datos continetur, atque ad partem tantum curvæ inter puncta data A & A' comprehenſam pertinet. Quoti ſcilicet $\frac{S}{P}$, $\frac{M}{P}$, quatenus curva intra hos limites continetur, tanquam immmutati ſpectantur.

§. 207. Figura propoſita, ſive rectilinea, ſive curvilinea, referatur ad punctum aliquod tanquam focum. Inveſtigatio maximi minimive eodem fere modo inſtituitur.

Problema. Sit S punctum poſitione datum; poſitione pariter dentur duo puncta A & A'. Centro S radiis SA, SA' deſcribantur circuli, qui rectæ cuicunque per S ductæ in B & B' occurrant. Centro S, radio quolibet SP inter SB, SB' medio, deſcribatur circulus. Sint F, F' coëfficientes magnitudine dati. In circumferentia radii SP aſſignandum ſit M punctum ſic, ut ſumma $F\times AM+F'\times A'M$ ſit omnium minima. Fig. 62.

Sint $SA=a$, $SA'=a'$, $SM=b$, $ASM=y$, $A'SM'=y'$, $AM=z$, $A'M=z'$.

Erit ideo $F\times z + F'\times z' =$ minimo;

hinc $F\times\frac{dz}{dv}+F'\times\frac{dz'}{dv} = 0$.

Atqui

Atqui $\begin{matrix} zz = aa - 2ab\,\text{coſ}.y + bb \\ z'z' = a'a' - 2a'b\,\text{coſ}.y' + bb \end{matrix}$;

ideoque
$$\frac{dz}{dv} = \frac{ab\,\text{ſin}.y\,\frac{dy}{dv}}{z} = a\,\text{ſin}.SAM\,\frac{dy}{dv}$$
$$\frac{dz'}{dv} = \frac{a'b\,\text{ſin}.y'\,\frac{dy'}{dv}}{z'} = b\,\text{ſin}.SMA'\,\frac{dy'}{dv}.$$

Quare $F \times a\,\text{ſin}.SAM \times \frac{dy}{dv} + F' \times b\,\text{ſin}.SMA' \times \frac{dy'}{dv} = 0.$

Sed $y + y' = ASA' =$ dato; ergo $\frac{dy}{dv} + \frac{dy'}{dv} = 0.$ Proinde $F \times a\,\text{ſin}.SAM = F' \times b\,\text{ſin}.SMA'$.

Fig. 63. Recta BB' dividatur in partes quotcunque æquales in punctis P, P', P'', P''', P^{v}... Centro S, radiis $SP, SP', SP'', SP''', SP^{iv}$... deſcribantur circuli; ſint F, F', F'', F''', F^{v}... totidem coëfficientes dati. Ut ſumma $F \times AM + F' \times MM' + F'' \times M'M'' + F''' \times M''M''' + F^{v} \times M'''M^{iv} + \ldots$ ſit omnium minima, debet eſſe
$$\begin{aligned} F \times SA\,\text{ſin}.SAM &= F' \times SM\,\text{ſin}.SMM' \\ &= F'' \times SM'\,\text{ſin}.SM'M'' \\ &= F''' \times SM''\,\text{ſin}.SM''M''' \\ &= F^{iv} \times SM'''\,\text{ſin}.SM'''M'' \\ &= - \quad - \quad - \quad - \end{aligned}$$

Hinc pro vertice quolibet M, duos inter $'M$, M' poſito, eſt $F \times SM \times \text{ſin}.SMM'$ quantitas conſtans.

Ab foco S concipiatur acta SQ ipſi MM perpendicularis: erit $F \times SQ$ quantitas conſtans.

Quoniam $F \times SM \times \text{ſin}.SMM'$ eſt quantitas conſtans; limes etiam quantitatis hujus eſt quantitas conſtans: proinde, aucto partium numero, in curva, quæ limes eſt figuræ rectilineæ $AMM'M''M'''$... per M ducta tangente MT, in quam demittatur perpendiculum SQ, eſt $F \times SM\text{ſin}.SMT = F \times SQ$ quantitas conſtans.

Poſito igitur $\frac{dZ}{dx} = \varphi x . \frac{dz}{dx}$, denotante φx functionem aliquam diſtantiæ FM:

curva,

curva, in qua Z eſt minima, ea eſt, ad quam $\varphi x \times SM\,\text{ſin.}SMT$, ſeu $\varphi x \times SQ$ eſt quantitas conſtans.

Scholium. Exemplum hoc refertur ad minimum quoddam, quod in motu projectilium circa centrum aliquod revolutorum locum habet. Nempe quotiescunque velocitas horum projectilium eſt functio aliqua diſtantiæ eorum a centro revolutionis, quæ dicatur V; facto $\frac{dZ}{dx} = V\frac{dz}{dx}$, eſt Z in curva deſcripta omnium minimum. Etenim $V \times SQ$ eſt quantitas conſtans per legem arearum a Keplero detectam, & a Newtono demonſtratam. (Vid. Euleri *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes. Additamentum 2.*)

§. 208. In exemplo præcedente ponebam, radios vectores æqualibus differentiis creſcere. Inveſtigatio $\frac{\text{maximi}}{\text{minimi}}$ fit eodem modo, poſito angulos ad focum æqualibus differentiis variare.

Angulus conſtans ASM dicatur α; & debeat eſſe $Fz+F'z'+F''z''+F'''z'''+F^{\text{iv}}z^{\text{iv}}+\ldots$ omnium $\frac{\text{maxima}}{\text{minima}}$. Fig. 64.

Hinc $F\frac{dz}{dv} + F'\frac{dz'}{dv} + F''\frac{dz''}{dv} + F'''\frac{dz'''}{dv} + F^{\text{iv}}\frac{dz^{\text{iv}}}{dv} + \ldots = 0.$

Radii SA, SM, SM', SM'', SM''', SM^{iv} . . .
dicantur a, y, y', y'', y''', y^{iv} . . . . reſpective.

Quoniam $zz = aa\,\text{ſin.}^2\alpha + (y - a\,\text{coſ.}\alpha)^2$ fit $\frac{dz}{dv} = \frac{dy}{dv}\,\text{coſ.}\,SMA$

$$z'z' = yy\,\text{ſin.}^2\alpha + (y' - y\,\text{coſ.}\alpha)^2 \qquad \frac{dz'}{dv} = \frac{dy'}{dv}\,\text{coſ.}SM'M - \frac{dy}{dv}\,\text{coſ.}(\alpha + SM'M)$$

$$z''z'' = y'y'\,\text{ſin.}^2\alpha + (y'' - y'\,\text{coſ.}\alpha)^2 \qquad \frac{dz''}{dv} = \frac{dy''}{dv}\,\text{coſ.}SM''M' - \frac{dy'}{dv}\,\text{coſ.}(\alpha + SM''M')$$

$$z'''z''' = y''y''\,\text{ſin.}^2\alpha + (y''' - y''\,\text{coſ.}\alpha)^2 \qquad \frac{dz'''}{dv} = \frac{dy'''}{dv}\,\text{coſ.}SM'''M'' - \frac{dy''}{dv}\,\text{coſ.}(\alpha + SM'''M'')$$

$$z^{\text{iv}}z^{\text{iv}} = y'''y'''\,\text{ſin.}^2\alpha + (y^{\text{iv}} - y'''\,\text{coſ.}\alpha)^2 \qquad \frac{dz^{\text{iv}}}{dv} = \frac{dy^{\text{iv}}}{dv}\,\text{coſ.}SM^{\text{iv}}M''' - \frac{dy'''}{dv}\,\text{coſ.}(\alpha + SM^{\text{iv}}M''')$$

\- \- \- \- \- \- \- \- \- \-

\- \- \- \- \- \- \- \- \- \-

unde

$$\text{unde}\quad \left.\begin{aligned}&\frac{dy}{dv}(F\,\text{coſ.}SMA - F'\text{coſ.}(\alpha+SM'M))\\ +&\frac{dy'}{dv}(F'\text{coſ.}SM'M - F''\text{coſ.}(\alpha+SM''M'))\\ +&\frac{dy''}{dv}(F''\text{coſ.}SM''M' - F'''\text{coſ.}(\alpha+SM'''M''))\\ +&\frac{dy'''}{dv}(F'''\text{coſ.}SM'''M'' - F^{\text{iv}}\text{coſ.}(\alpha+SM^{\text{iv}}M'''))\\ +&\;-\;-\;-\;-\;-\;-\;-\\ +&\;-\;-\;-\;-\;-\;-\;-\end{aligned}\right\} = 0.$$

Omnes quantitates mutabiles y, y', y'', y''', y^{iv} . . . fiant ſimul conſtantes, una excepta: erit pro vertice quolibet M, duobus $'M$, M' interjacente, $F\text{coſ.SM'M} = F'\text{coſ.}(\alpha+\text{SM'M}) = -F'\text{coſ.SMM'}$.

Exemplum. Sit $F = F' = F'' = F''' = F^{\text{iv}} \ldots$: erit $F\text{coſ.SM'M} = -F\text{coſ.SMM'}$; ideoque $\text{coſ.SM'M} = -\text{coſ.SMM'}$; $\text{SMM'} = 180^\circ - \text{SM'M}$: proinde puncta 'M, M, M' jacent in directum; & tota via $AMM'M''M'''$ . . . eſt linea recta.

Dividatur angulus AFA' in partes quotcunque invicem æquales, quarum quævis dicatur Δx; & ſit F' quantitas mutabilis ab x dependens, ita ut ſit
$$F' = F + \frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots$$

Erit $F\text{coſ.SM'M} = (F + \frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots)\,\text{coſ.}(\alpha+\text{SM'M})$;

unde

$$F(\text{coſ.SM'M} - \text{coſ.}(\alpha+\text{SM'M})) = \left(\frac{\Delta x}{1}\cdot\frac{dF}{dx} + \frac{\Delta x^2}{1.2}\cdot\frac{ddF}{dx^2} + \ldots\right)\text{coſ.}(\alpha+\text{SM'M})$$

$$F\frac{\text{coſ.SM'M} - \text{coſ.}(\alpha+\text{SM'M})}{\Delta x} = \left(\frac{dF}{dx} + \frac{\Delta x}{1.2}\cdot\frac{ddF}{dx^2} + \frac{\Delta x^2}{1.2.3}\cdot\frac{d^3F}{dx^3} + \ldots\right)\text{coſ.}(\alpha+\text{SM'M}).$$

Quare & membrorum æquationis hujus limites ſunt invicem æquales; nempe

$$-F\frac{d.\frac{dy}{dz}}{dx} = \frac{dF}{dx}\cdot\frac{dy}{dz}:$$

$$\text{unde}\quad \frac{dF}{dx}\cdot\frac{dy}{dz} + F\frac{d.\frac{dy}{dz}}{dx} = 0$$

$$F\frac{dy}{dz} = C.$$ Itaque res ad calculum integralem ſemper reducitur.

Ex æquatione $\frac{dy}{dz} = \frac{C}{F}$ deducitur $\frac{dx}{dy} = \frac{C}{\sqrt{(FF - CCyy)}}$.

§. 209.

§. 209. Transeo ad exempla, quibus factor F eſt functio aliqua ordinatarum.

Problema. Omnibus uti §. 202. poſitis, quæritur ſumma

$$z(a+y)+z'(y+y')+z''(y'+y'')+z'''(y''+y''')+\ldots=\frac{\text{maximo}}{\text{minimo}}.$$

Erit ideo $\frac{dy}{dv}(z+z'+(a+y)\,\text{coſ.}PMA-(y+y')\text{coſ.}P'M'M)$ Fig. 61.

$$+\frac{dy'}{dv}(z'+z''+(y+y')\,\text{coſ.}P'M'M-(y'+y'')\,\text{coſ.}P''M''M')$$

$$+\frac{dy''}{dv}(z''+z'''+(y'+y'')\,\text{coſ.}P''M''M'-(y''+y''')\,\text{coſ.}P'''M'''M'')$$

$$+\frac{dy'''}{dv}(z'''+z^{\text{iv}}+(y''+y''')\,\text{coſ.}P'''M'''M''-(y'''+y^{\text{iv}})\,\text{coſ.}P^{\text{iv}}M^{\text{iv}}M''')$$

+ - - - - - - -

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Omnes quantitates mutabiles y, y', y'', y''', $y^{\text{iv}}\ldots$ fiant ſimul conſtantes, una excepta: erit pro vertice quolibet M, inter duos vertices $'M$, M' comprehenſo, $\frac{dy}{dv}(z+z'+('y+y)\,\text{coſ.PM'M}-(y+y')\,\text{coſ.P'M'M}) = 0$.

Atqui $'y = y-\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}-\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\ldots$

$$y' = y+\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{1.2.3}\cdot\frac{d^3y}{dx^3}+\ldots$$

Igitur $z+z'+2y(\text{coſ.PM'M}-\text{coſ.P'M'M})-\frac{\Delta x}{1}\cdot\frac{dy}{dx}(\text{coſ.PM'M}+\text{coſ.P'M'M})$

$$+\frac{\Delta x^2}{1.2}\cdot\frac{ddy}{dx^2}(\text{coſ.PM'M}-\text{coſ.P'M'M}) = 0.$$

— - - - - - -

Proinde & limes ſemiſſis hujus quantitatis eſt zero; nempe

$$\frac{dz}{dx}-y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}-\frac{dy}{dx}\cdot\frac{dy}{dz}=0,\ \text{ſeu}\ \frac{1}{\frac{dz}{dx}}-y\cdot\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0:\ \text{ideoque}$$

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}}-y\times\frac{\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=0;\ \frac{y}{\frac{dz}{dx}}=C;\ \frac{dx}{dz}=\frac{C}{y}.$$ Unde res ad calculum integralem revocatur.

Sit nempe $\frac{dZ}{dx} = y\frac{dz}{dx}$: functio Z ſit omnium maxima, quando $\frac{dx}{dz} = \frac{C}{y}$.

Generatim. Sit $z(\varphi a+\varphi y)+z'(\varphi y+\varphi y')+z''(\varphi y'+\varphi y'')+z'''(\varphi y''+\varphi y''')+\ldots=$ maximo.

$$\left.\begin{array}{l}
\text{Erit } \frac{dy}{dv}(\pi y(z+z')+(\varphi a+\varphi y)\,\text{coſ.}\,PM'M-(\varphi y+\varphi y')\,\text{coſ.}\,P'M'M) \\
\quad + \frac{dy'}{dv}(\pi y'(z'+z'')+(\varphi y+\varphi y')\,\text{coſ.}\,P'M'M-(\varphi y'+\varphi y'')\,\text{coſ.}\,P''M''M') \\
\quad + \frac{dy''}{dv}(\pi y''(z''+z''')+(\varphi y'+\varphi y'')\,\text{coſ.}\,P''M''M'-(\varphi y''+\varphi y''')\,\text{coſ.}\,P'''M'''M'') \\
\quad + \frac{dy'''}{dv}(\pi y'''(z'''+z^{\text{IV}})+(\varphi y''+\varphi y''')\,\text{coſ.}\,P'''M'''M''-(\varphi y'''+\varphi y^{\text{IV}})\,\text{coſ.}\,P^{\text{IV}}M^{\text{IV}}M''') \\
\quad + \;\; - \;\; - \;\; - \;\; - \;\; - \;\; - \\
\quad + \;\; - \;\; - \;\; - \;\; - \;\; - \;\; -
\end{array}\right\} = 0.$$

Hinc poſitis omnibus quantitatibus mutabilibus y, y', y'', y''', y''', $y^{\text{IV}}\ldots$ ſimul conſtantibus, præter unam, fit pro vertice quolibet M, duobus 'M, M' interjacente, $\pi y(z+z')+(\varphi' y+\varphi y)\,\text{coſ.PM'M}-(\varphi y+\varphi y')\,\text{coſ.P'M'M} = 0$.

Atqui $\varphi' y = \varphi y - \frac{\Delta x}{1}\cdot \pi y\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2} + \ldots$

$$\varphi y' = \varphi y + \frac{\Delta x}{1}\cdot \pi y\frac{dy}{dx} + \frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2} + \ldots$$

Ergo

$$\left.\begin{array}{l}
\pi y\left(\frac{z+z'}{\Delta x}\right) + \varphi y\frac{\text{coſ.PM'M} - \text{coſ.P'M'M}}{\Delta x} - \pi y\frac{dy}{\Delta x}(\text{coſ.PM'M}+\text{coſ.P'M'M}) \\
\qquad\qquad + \frac{\Delta x}{1.2}\frac{d^2\varphi y}{dx^2}(\text{coſ.PM'M}-\text{coſ.P'M'M}) \\
\qquad\qquad - \;\; - \;\; - \;\; - \;\; -
\end{array}\right\} = 0.$$

Unde & limes dimidiæ hujus quantitatis $= 0$; nempe

$$\pi y\frac{dz}{dx} - \varphi y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} - \pi y\frac{dy}{dx} - \frac{dz}{dx} = 0$$

$$\pi y\frac{1}{\frac{dz}{dx}} - \varphi y\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$$

πy

$$\pi y\,\frac{\frac{dy}{dx}}{\frac{dz}{dx}} - \varphi y\,\frac{\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$$

tandemque $\varphi y\,\frac{1}{\frac{dz}{dx}} = C$, & $\frac{dx}{dz} = \frac{C}{\varphi y}$.

In curva igitur, ad quam eſt $\frac{dZ}{dx} = \frac{dz}{dx}\varphi y$, Z eſt omnium maxima, quando $\frac{dx}{dz} = \frac{C}{\varphi y}$.

§. 210. Inveſtigatio eodem modo inſtituitur in curvis ad focum aliquem relatis.

Sint nempe y, y', y'', y''', y^{IV} ... radii vectores, quos inter angulus conſtans α comprehenditur; & debeat eſſe

$$z(\varphi a+\varphi y)+z'(\varphi y+\varphi y')+z''(\varphi y'+\varphi y'')+z'''(\varphi y''+\varphi y''')+z^{IV}(\varphi y'''+\varphi y^{IV})+\ldots = \text{max.}$$

Erit

$$\left.\begin{array}{l}
\frac{dy}{dv}(\pi y(z+z')+(\varphi a+\varphi y)\,\text{coſ.}SMA-(\varphi y+\varphi y')\,\text{coſ.}(\alpha+SM'M)) \\
+\frac{dy'}{dv}(\pi y'(z'+z'')+(\varphi y+\varphi y')\,\text{coſ.}SM'M-(\varphi y'+\varphi y'')\,\text{coſ.}(\alpha+SM''M')) \\
+\frac{dy''}{dv}(\pi y''(z''+z''')+(\varphi y'+\varphi y'')\,\text{coſ.}SM''M'-(\varphi y''+\varphi y''')\,\text{coſ.}(\alpha+SM'''M'')) \\
+\frac{dy'''}{dv}(\pi y'''(z'''+z^{IV})+(\varphi y''+\varphi y''')\,\text{coſ.}SM'''M''-(\varphi y'''+\varphi y^{IV})\,\text{coſ.}(\alpha+SM^{IV}M''')) \\
+\;-\;-\;-\;-\;-\;-\;- \\
+\;-\;-\;-\;-\;-\;-
\end{array}\right\} = 0.$$

Unde pro quolibet vertice M, duos inter $'M$, M' comprehenſo, eſt

$$\pi y(z+z')+(\varphi' y+\varphi y)\,\text{coſ.}SM'M-(\varphi y+\varphi y')\,\text{coſ.}SM'M) = 0.$$

Hinc

$$\left.\begin{array}{l}
\pi y(z+z')+2\varphi y(\text{coſ.}SM'M-\text{coſ.}(\alpha+SM'M))-\frac{\Delta x}{1}.\pi y\frac{dy}{dx}(\text{coſ.}SM'M+\text{coſ.}(\alpha+SM'M)) \\
\qquad+\frac{\Delta x^2}{1.2}\frac{d^2\varphi y}{dx^2}(\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;) \\
\qquad+\;-\;-\;-\;-\;-\;-
\end{array}\right\} = 0.$$

Ideo-

Ideoque etiam

$$\pi y \frac{z+z'}{\Delta x} + 2\varphi y \frac{\text{coſ.SM'M} - \text{coſ.}(\alpha+\text{SM'M})}{\Delta x} - \pi y \frac{dy}{dx}(\text{coſ.SM'M} + \text{coſ.}\alpha + \text{SM'M}) + \frac{\Delta x}{1.2}\frac{d^2\varphi y}{dx^2}(\ldots\ldots\ldots) = 0.$$

Ideo & limes quantitatis hujus eſt zero.

Atqui $\lim.\frac{\text{coſ.SM'M-coſ.}(\alpha+\text{SM'M})}{\Delta x} = \frac{-\text{d}.\frac{dy}{dz}}{dx} = \frac{y\left(\frac{dy}{dx}\right)^2 - yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}$.

Ergo $\pi y\left(\frac{dz}{dx} - \frac{dy}{dx}.\frac{dy}{dz}\right) + \varphi y \frac{y\left(\frac{dy}{dx}\right)^2 - yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$

ſeu $\pi y \frac{yy}{\frac{dz}{dx}} + \varphi y \frac{y\left(\frac{dy}{dx}\right)^2 - yy\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$

$$\pi y \frac{y\frac{dy}{dx}}{\frac{dz}{dx}} + \varphi y \frac{\left(\frac{dy}{dx}\right)^3 - y\frac{dy}{dx}.\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = 0$$

Unde $\varphi y \frac{y}{\frac{dz}{dx}} = C$, & $\frac{C}{\varphi y} = y\frac{dx}{dz} = \text{ſin.}FMT.$

Hactenus dicta pertinent ad methodum $\genfrac{}{}{0pt}{}{\text{maximorum}}{\text{minimorum}}$ abſolutam; ſeu eam, in qua curvis aut figuris propoſitis nulla proprietas communis tribuitur. Transeo ad exempla, quibus non inter omnes abſolute figuras vel curvas, ſed eas inter, quæ data quadam proprietate communi gaudeant, quæritur figura vel curva, $\genfrac{}{}{0pt}{}{\text{maximi}}{\text{minimi}}$ proprietate aliqua prædita.

Fig. 65. §. 211. *Problema.* Sint A, A' duo puncta poſitione data, relata ad rectam BB' per demiſſa in eam perpendicula AB, $A'B'$. Axi BB', in tres partes æquales diviſo in punctis P, P', perpendiculares in his punctis conſtituantur rectæ PM, $P'M'$. Quæruntur earum puncta M & M': ut ſumma rectarum AM, MM', $M'A'$ detur magnitudine; & ſpatium $BAMM'B'A'$ ſit omnium maximum.

Sint

Sint $AB = a$, $A'B' = a'$, $BP = b$, $MP = y$, $M'P' = y'$, $AM = z$, $MM' = z'$, $M'A' = z''$.

Erit ideo $\begin{array}{rl} z + z' + z'' & = \text{dato} \\ b(a+2y+2y'+a') & = \text{maximo} \end{array}$ ſeu $\begin{array}{rl} z + z' + z'' & = \text{dato} \\ y + y' & = \text{maximo.} \end{array}$

Hinc $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} = 0$

$$\frac{dy}{dv} + \frac{dy'}{dv} = 0.$$

Atqui $\frac{dz}{dv} = \frac{dy}{dv} \operatorname{coſ}.MAB$

$$\frac{dz'}{dv} = \left(\frac{dy'}{dv} - \frac{dy}{dv}\right) \operatorname{coſ}.M'MP$$

$$\frac{dz''}{dv} = -\frac{dy'}{dv} \operatorname{coſ}.A'M'P'.$$

Proinde $\frac{dy}{dv}(\operatorname{coſ}.MAB - \operatorname{coſ}.M'MP) + \frac{dy'}{dv}(\operatorname{coſ}.M'MP - \operatorname{coſ}.A'M'P') = 0$

$$\frac{dy}{dv} + \frac{dy'}{dv} = 0.$$

Hinc $\operatorname{coſ}.MAB - \operatorname{coſ}.M'MP = \operatorname{coſ}.M'MP - \operatorname{coſ}.A'M'P'$.

Dividatur axis BB' in partes quotcunque æquales, quarum quævis dicatur Δx, in P, P', P'', P''', P^{iv}... punctis; ex quibus agantur rectæ ipſi BB' perpendiculares. Summa $z + z' + z'' + z''' + z^{iv} + \ldots$ magnitudine data; maxima eſt ſumma $\Delta x(a + 2y + 2y' + 2y'' + 2y''' + 2y^{iv} + \ldots)$, quando Fig. 61.

$$\begin{aligned} \operatorname{coſ}.MAB - \operatorname{coſ}.M'MP &= \operatorname{coſ}.M'MP - \operatorname{coſ}.M''M'P' \\ &= \operatorname{coſ}.M''M'P' - M'''M''P'' \\ &= \operatorname{coſ}.M'''M''P'' - \operatorname{coſ}.M^{iv}M'''P''' \\ &= - \quad - \quad - \quad - \quad - \\ &= - \quad - \quad - \quad - \quad - \end{aligned}$$

Pro quolibet igitur vertice M, duos inter 'M, M' poſito, eſt $\operatorname{coſ}.M'M'P - \operatorname{coſ}.M'MP$ quantitas conſtans, manente Δx eadem; proinde & $\operatorname{coſ}.\frac{M'M'P - M'MP}{\Delta x}$, paritérque limes ejus, hoc eſt (§. 192. *Obſ.* 3.), $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}$ eſt quantitas conſtans. Quare

inter

inter curvas, data perimetro terminatas, ea maximam comprehendit aream, cujus radius curvaturæ eſt datæ magnitudinis; ac proinde curva hæc eſt circumferentia circuli, uti methodo mere elementari demonſtrari poteſt.

Quoniam $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \frac{1}{C}$: eſt $\frac{-\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \frac{dy}{dx}\cdot\frac{1}{C}$; hinc $\frac{1}{\frac{dz}{dx}} = \frac{C'+y}{C}$,

$\frac{dx}{dz} = \frac{C'+y}{C} = \text{ſin.}PMT.$

Hinc deducitur $\frac{dx}{dy} = \frac{C'+y}{\sqrt{(CC-(C'+y)^2)}}$; $x = C'' - \sqrt{(CC-(C'+y)^2)}$: unde $CC = (C''-x)^2 + (C'-y)^2$, quæ eſt etiam æquatio circumferentiæ circuli.

§. 212. *Problema.* Perimetro $AMM'A'$ magnitudine data; requiritur, ut ſolidum gyratione figuræ $BAMM'A'B'$ circa axem BB' genitum ſit omnium maximum.

Erit ideo $z + z' + z'' = \text{dato}$

$b((aa+ay+yy)+(yy+yy'+y'y')+(y'y'+y'a'+a'a')) = \text{maximo}$

ſeu $z + z' + z'' = \text{dato}$

$ay + 2yy + yy' + 2y'y' + y'a' = \text{maximo.}$

Hinc $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} = 0$

ſeu $\frac{dy}{dv}(\text{coſ.}MAB - \text{coſ.}M'MP) + \frac{dy'}{dv}(\text{coſ.}M'MP - \text{coſ.}A'M'P') = 0$

& $\frac{dy}{dv}(a+4y+y') + \frac{dy'}{dv}(y+4y'+a') = 0.$

Quare $\frac{\text{coſ.}MAB - \text{coſ.}M'MP}{a+4y+y'} = \frac{\text{coſ.}M'MP - \text{coſ.}A'M'P'}{y+4y'+a'}.$

Dividatur axis BB' in partes quotcunque æquales in P, P', P'', P''', P^{iv} ... punctis: erit pro dato partium numero, & pro vertice quolibet M, duos inter 'M, M' comprehenſo, $\frac{\text{coſ.}M'M'P - \text{coſ.}M'MP}{'y+4y+y'}$ quantitas conſtans; proinde & quan-

quantitas $\frac{\text{coſ.}M'M'P-\text{coſ.}M'MP}{\Delta x('y+4y+y')}$, pariterque ejus limes eſt quantitas conſtans. Atqui limes hic eſt $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}\times\frac{1}{6y}$. In curva igitur longitudine data, quæ circa axem BB' rotata maximum generat ſolidum, eſt $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}\times\frac{1}{y}$ quædam quantitas conſtans; ſeu rectangulum ſub recta axi huic ordinatim applicata & ſub radio curvaturæ ſemper eſt idem.

Sit $\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}\times\frac{1}{y}=\frac{2}{CC}$: erit $\frac{-\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = -\frac{2y\frac{dy}{dx}}{CC}$

$$\text{hinc}\ \frac{1}{\frac{dz}{dx}} = \frac{C'C'-yy}{CC} = \frac{dx}{dz} = \text{ſin.}PMT.$$

Unde res ad calculum integralem reducitur.

Scholium. Curva problemate hoc determinata dicitur *elaſtica.*

§. 213. *Generatim.* Sit $z+z'+z''+z'''+z^{\text{iv}}+\ldots$ magnitudine data; & ſumma $(\varphi a+\varphi y)+(\varphi y+\varphi y')+(\varphi y'+\varphi y'')+(\varphi y''+\varphi y''')+\ldots$ ſeu ſumma $\varphi y+\varphi y'+\varphi y''+\varphi y'''+\varphi y^{\text{iv}}+\ldots$ debeat eſſe omnium maxima.

Erit $\frac{dz}{dv}+\frac{dz'}{dv}+\frac{dz''}{dv}+\frac{dz'''}{dv}+\frac{dz^{\text{iv}}}{dv}+\ldots=0$

$$\pi y\frac{dy}{dz}+\pi y'\frac{dy'}{dv}+\pi y''\frac{dy''}{dv}+\pi y'''\frac{dy'''}{dv}+\pi y^{\text{iv}}\frac{dy^{\text{iv}}}{dv}+\ldots=0.$$

Unde $\frac{dy}{dv}(\text{coſ.}PMA-\text{coſ.}P'M'M)$

$+\frac{dy'}{dv}(\text{coſ.}P'M'M-\text{coſ.}P''M''M')$

$+\frac{dy''}{dv}(\text{coſ.}P''M''M'-\text{coſ.}P'''M'''M'')$ $=0.$

$+\frac{dy'''}{dv}(\text{coſ.}P'''M'''M''-\text{coſ.}P^{\text{iv}}M^{\text{iv}}M''')$

$+$ - - - - - -

Omnes

Omnes quantitates mutabiles y, y', y'', y''', y''''... fiant ſimul conſtantes, præter duas. Erit $\frac{\text{coſ.PMA} - \text{coſ.P}'\text{M}'\text{M}}{\pi y} = \frac{\text{coſ.P}'\text{M}'\text{M} - \text{coſ.P}''\text{M}''\text{M}'}{\pi y'}$

$$= \frac{\text{coſ.P}''\text{M}''\text{M}' - \text{coſ.P}'''\text{M}'''\text{M}''}{\pi y''}$$

$$= \frac{\text{coſ.P}'''\text{M}'''\text{M}'' - \text{coſ.P}^{\text{IV}}\text{M}^{\text{IV}}\text{M}'''}{\pi y'''}$$

$$= - \quad - \quad - \quad - \quad -$$

Hinc pro vertice quolibet M, duobus 'M, M' interjacente, eſt $\frac{\text{coſ.PM}'\text{M} - \text{coſ.P}'\text{M}'\text{M}}{\pi y}$ quantitas conſtans, a quantitate Δx pendens; proinde & quantitas $\frac{\text{coſ.PM}'\text{M} - \text{coſ.P}'\text{M}'\text{M}}{\Delta x . \pi y}$, pariterque limes ejus, nempe $\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3 \pi y}$ eſt quantitas conſtans C.

Hinc $\frac{-\frac{dy}{dx} \cdot \frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y \frac{dy}{dx}$

ideoque $\frac{1}{\frac{dz}{dx}} = C' + C\varphi y$; ſeu $\frac{dx}{dz} = C' + C\varphi y = \text{ſin.}PMT$. Unde res ad calculum integralem reducitur.

In curva igitur, cujus perimeter magnitudine datur, ſi fuerit $\frac{dZ}{dx} = \varphi y$; functio Z ſit omnium maxima, quando radius curvaturæ eſt in ratione inverſa quantitatis πy ſeu $\frac{d\varphi y}{dx}$, ſeu quando ſin.$PMT = \frac{dx}{dz} = C' + C\varphi y$.

§. 214. Omnibus uti §. 211. poſitis, requiritur, ut momentum figuræ $BAMM'A'B'$ reſpectu lineæ DD' ipſi BB' perpendicularis ſit omnium maximum.

Fig. 66. Sint AD, $A'D'$, MQ, $M'Q$ ipſi DQ' perpendiculares.

Lemma. Momentum trapezii $BAMP$ reſpectu rectæ DD', lateribus ejus AB, MP parallelæ, proportionale eſt ſummæ $(AB(2AD+MQ)+MP(2MQ+AD))\,BP$.

Sint $AD = b$, $A'D' = b'$, $MQ = c$, $M'Q' = c'$.

Igitur $z + z' + z'' = \text{dato}$

$a(2b+c)+y(2c+b)+y(2c+c')+y'(2c'+c)+y'(2c'+b')+a'(2b'+c) = \text{maximo}$,

ſeu $y(b+4c+c')+y'(c+4c'+b') = \text{max.}$

Hinc

$$\text{Hinc } \frac{dy}{dv}(\text{coſ.PMA}-\text{coſ.P'M'M}) + \frac{dy'}{dv}(\text{coſ.P'M'M}-\text{coſ.BA'M'}) = 0$$

$$\frac{dy}{dv}(b+4c+c') + \frac{dy'}{dv}(c+4c'+b') = 0.$$

$$\text{Ideoque } \frac{\text{coſ.PMA}-\text{coſ.P'M'M}}{b+4c+c'} = \frac{\text{coſ.P'M'M}-\text{coſ.B'A'M'}}{c+4c'+b'}.$$

Hinc axe BB' diviſo in partes quotcunque æquales, quarum quælibet ſit Δx; & poſita $SP = x$: erit pro vertice quolibet M, duos inter 'M, M' ſito, $\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{6x\Delta x}$ quantitas conſtans; quare etiam $\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{x\Delta x}$, Fig. 61.

atque ejus limes eſt quædam conſtans, nempe $\frac{-\frac{ddy}{dx^2}}{x\left(\frac{dz}{dx}\right)^3} = C$: ac proinde difficultas curvam propoſitam determinandi pendet ab imperfectione calculi integralis.

Eadem difficultas ſe offert: ſi, data perimetro, fuerit $\frac{dZ}{dx} = Fx \times \varphi y$; & debeat eſſe Z omnium maxima. Erit enim $\frac{-\frac{ddy}{dx^2}}{Fx\cdot\left(\frac{dz}{dx}\right)^3 \pi y} = C$, ſeu

$\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = CF \times \pi y$, cujus integratio in terminis finitis deſideratur.

§. 215. Quodſi figuræ, ſive rectilineæ, ſive curvilineæ, ad punctum aliquod referuntur; modus procedendi non erit multum diverſus.

Sit S focus, ad quem figura refertur per radios vectores SA, SM, SM', SM'', SM'''... Data perimetro $AMM'M''M'''$... figura $SAMM'M''M'''$... debeat eſſe omnium maxima. Sint anguli ASM, MSM', $M'SM''$, $M''SM'''$... invicem æquales, qui dicantur α. Fig. 64.

Erit $z + z' + z'' + z''' + z^{iv} + \ldots =$ dato

$\text{ſin.}\alpha(ay + yy' + y'y'' + y''y''' + y'''y^{iv} + \ldots =$ maximo.

$$\text{Hinc ſit } \frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{iv}}{dv} + \ldots = 0$$

 ſeu

$$\text{ſeu}\quad \frac{dy}{dv}\left(\text{coſ.}SMA - \text{coſ.}(\alpha + SM'M)\right)$$
$$+ \frac{dy'}{dv}\left(\text{coſ.}SM'M - \text{coſ.}(\alpha + SM''M')\right)$$
$$+ \frac{dy''}{dv}\left(\text{coſ.}SM''M' - \text{coſ.}(\alpha + SM'''M'')\right) \qquad = 0$$
$$+ \frac{dy'''}{dv}\left(\text{coſ.}M'''M'' - \text{coſ.}(\alpha + SM^{iv}M''')\right)$$
$$+ \;-\;-\;-\;-\;-\;-$$

$$\text{Item}\quad \frac{dy}{dv}(a+y') + \frac{dy'}{dv}(y+y') + \frac{dy''}{dv}(y'+y'') + \frac{dy'''}{dv}(y''+y''') + \ldots = 0$$

Omnibus igitur quantitatibus mutabilibus y, y', y'', y''', y^{iv} . . . præter duas, poſitis conſtantibus, fit
$$\frac{\text{coſ.}SMA - \text{coſ.}(\alpha + SM'M)}{a+y'} = \frac{\text{coſ.}SM'M - \text{coſ.}(\alpha + SM''M')}{y+y'}$$
$$= \frac{\text{coſ.}SM''M' - \text{coſ.}(\alpha + SM'''M'')}{y'+y''}$$
$$= \frac{\text{coſ.}SM'''M'' - \text{coſ.}(\alpha + SM^{iv}M''')}{y''+y'''}$$
$$= \;-\;-\;-\;-\;-$$

Quare pro vertice quovis M, duos inter $'M$, M' jacente, fit $\dfrac{\text{coſ.}SMM' - \text{coſ.}(\Delta x + SM'M)}{\Delta x('y+y')}$ quantitas conſtans: proinde & limes quantitatis hujus duplæ conſtans eſt, nempe $\dfrac{yy + 2\left(\frac{dy}{dx}\right)^2 - y\,\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = \dfrac{1}{C}$; cui æquationi ſatisfacit circumferentia circuli (§. 196.).

§. 216. *Generatim* ſit
$$z + z' + z'' + z''' + z^{iv} + \ldots = \text{dato}$$
$$\varphi y + \varphi y' + \varphi y'' + \varphi y''' + \varphi y^{iv} + \ldots = \text{maximo}.$$
Fit $\dfrac{\text{coſ.}SM'M - \text{coſ.}(\alpha + S'M'M)}{\Delta x.\pi y} = C$; & limes hujus quantitatis, nempe
$$\frac{y^3 + 2y\left(\frac{dy}{dx}\right)^2 - yy\frac{ddy}{dx^2}}{\pi y\left(\frac{dz}{dx}\right)^3} = C; \;\&\; \frac{y\left(yy + 2\left(\frac{dy}{dx}\right)^2 - y\frac{ddy}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^3} = C\pi y.$$
Proinde radius curvaturæ eſt in ratione directa radii vectoris, & inverſa functionis πy.

§. 217. Curva referatur ad aliquem axem per coordinatas rectilineas; & proponantur duæ æquationes

$$zFy + z'Fy' + z''Fy'' + z'''Fy''' + z^{\text{IV}}Fy^{\text{IV}} + \dots = \text{dato}$$

$$\varphi y + \varphi y' + \varphi y'' + \varphi y''' + \varphi y^{\text{IV}} + \dots = \text{maximo}.$$

Poſito $\frac{dFy}{dx} = fy\frac{dy}{dv}$,

erit $\frac{dy}{dv}(zfy + Fy\,\text{coſ.}PM'M - Fy'\text{coſ.}P'M'M)$ Fig. 61.

$$\left.\begin{array}{l} \frac{dy}{dv}(zfy + Fy\,\text{coſ.}PM'M - Fy'\text{coſ.}P'M'M) \\ + \frac{dy'}{dv}(z'fy' + Fy'\text{coſ.}P'M'M - Fy''\text{coſ.}P''M''M') \\ + \frac{dy''}{dv}(z''fy'' + Fy''\text{coſ.}P''M''M' - Fy'''\text{coſ.}P'''M'''M'') \\ + \frac{dy'''}{dv}(z'''fy''' + Fy'''\text{coſ.}P'''M'''M'' - Fy^{\text{IV}}\text{coſ.}P^{\text{IV}}M^{\text{IV}}M''') \\ + \; - \quad - \quad - \quad - \quad - \end{array}\right\} = 0$$

Et ſimul $\frac{dy}{dv}\cdot\pi y + \frac{dy'}{dv}\cdot\pi y' + \frac{dy''}{dv}\cdot\pi y'' + \frac{dy'''}{dv}\cdot\pi y''' + \dots = 0$

Hinc fit $\frac{zfy + Fy\,\text{coſ.}PM'M - Fy'P'M'M}{\pi y} = C.$

Atqui $Fy' = Fy + \frac{\Delta x}{1} fy + \frac{\Delta x^2}{1.2}\frac{d^2Fy}{dx^2} + \frac{\Delta x^3}{1.2.3}\frac{d^3Fy}{dx^3} + \dots$

Quare $\frac{\frac{z}{\Delta x}fy + Fy(\text{coſ.}PM'M - \text{coſ.}P'M'M) - fy\frac{dy}{dx}\text{coſ.}P'M'M - \frac{\Delta x}{1.2}\frac{d^2Fy}{dx^2}\text{coſ.}P'M'M - \dots}{\pi y} = C.$

Quare & limes hujus quantitatis eſt quantitas conſtans: unde fit

$$fy\cdot\frac{1}{\frac{dz}{dx}} - Fy\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y$$

$$fy\cdot\frac{\frac{dy}{dx}}{\frac{dz}{dx}} - Fy\frac{\frac{dy}{dx}\cdot\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C\pi y\cdot\frac{dy}{dx}$$

$$Fy : \frac{1}{\frac{dz}{dx}} = C' + C\varphi y$$

$$\frac{dx}{dz} = \frac{C' + C\varphi y}{Fy} = \text{ſin.}PMT.$$

Unde res ad calculum integralem redit.

Sit

Sit nempe Z functio aliqua communis ejusmodi, ut $\frac{dZ}{dx} = \frac{dz}{dx}Fy$;

et ſit Z' alia quædam functio talis, ut $\frac{dZ'}{dx} = \varphi y$:

Functio Z' ſit omnium maxima, quando $\frac{dx}{dz} = \frac{C' + C\varphi y}{Fy}$.

Exemplum primum. Superficies ſolidi rotatione curvæ alicujus circa axem BB' geniti detur magnitudine; & area curvæ genitricis debeat eſſe omnium maxima.

Erit $\left.\begin{array}{l}\frac{dZ}{dx} = y\frac{dz}{dx} \\ \frac{dZ'}{dx} = y\end{array}\right.$: hinc $\frac{dx}{dz} = \frac{Cy \pm C'}{y}$; quæ eſt æquatio ad lineam rectam caſu, quo $C' = 0$.

Exemplum ſecundum. Superficies ſolidi rotatione curvæ alicujus circa axem BB' geniti detur magnitudine; & capacitas ſolidi debeat eſſe omnium maxima.

Erit $\left.\begin{array}{l}\frac{dZ}{dx} = y\frac{dz}{dx} \\ \frac{dZ'}{dx} = yy\end{array}\right.$: $\frac{dx}{dz} = \frac{Cyy \pm C'}{y}$; quæ eſt æquatio ad circumferentiam circuli caſu, quo $C' = 0$.

§. 218. *Problema.* Inter figuras ejusdem perimetri eam determinare, cujus centrum gravitatis ſit omnium maxime depreſſum reſpectu alicujus rectæ axi ejus perpendicularis.

Itaque eſt $z + z' + z'' + z''' + z^{IV} + \ldots = \text{dato}$

$$\frac{3M}{S} = \frac{a(2b+c)+y(b+4c+c')+y'(c+4c'+c'')+y''(c'+4c''+c''')+y'''(c''+4c'''+c^{IV})+\ldots}{a + 2y + 2y' + 2y'' + 2y''' + 2y^{IV} + \ldots} = \text{max.}$$

Unde $\frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{IV}}{dv} + \ldots = 0$

$$S\Big(\frac{dy}{dv}(b+4c+c') + \frac{dy'}{dv}(c+4c'+c'') + \frac{dy''}{dv}(c'+4c''+c''') + \frac{dy'''}{dv}(c''+4c'''+c^{IV}) + \ldots$$

$$= 6M\Big(\frac{dy}{dv} + \frac{dy'}{dv} + \frac{dy''}{dv} + \frac{dy'''}{dv} + \ldots\Big)$$

Hinc

$$\text{Hinc}\quad \frac{dy}{dv}(\text{coſ.}PMA - \text{coſ.}P'M'M)$$
$$+\frac{dy'}{dv}(\text{coſ.}P'M'M - \text{coſ.}P''M''M')$$
$$+\frac{dy''}{dv}(\text{coſ.}P''M''M' - \text{coſ.}P'''M'''M'') \quad = 0$$
$$+\frac{dy'''}{dv}(\text{coſ.}P'''M'''M'' - \text{coſ.}P^{IV}M^{IV}M''')$$
$$+ \text{ - - - - - -}$$

Igitur $\dfrac{\text{coſ.PM'M} - \text{coſ.P'M'M}}{S('c+4c+c') - 6\text{M}} = C$

pariterque lim. $\dfrac{\dfrac{\text{coſ.PM'M} - \text{coſ.P'M'M}}{\Delta x}}{S('x+4x+x') - 6\text{M}} = C$; ſeu $\dfrac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3} = C(Sx - M)$:

unde $\dfrac{dy}{dz} = C(\tfrac{1}{2}Sxx - Mx - C')$.

Hoc exemplo ad curvam translato, fit $\dfrac{dZ}{dx} = \dfrac{dz}{dx}$, $\dfrac{dM}{dx} = xy$, $\dfrac{dS}{dx} = y$: & poſita Z magnitudine data, $\dfrac{M}{S}$ fit omnium maxima, quando $\dfrac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3 (Sx-M)} = C$.

Eodem modo tractari poſſunt aliæ formulæ, quibus fit $\dfrac{M}{S}$ = maxima, poſitis $\begin{matrix}\dfrac{dM}{dx} = Fx \cdot \varphi y \\ \dfrac{dS}{dx} = \varphi' y\end{matrix}$. Sed imperfecta calculi integralis conditio obſtat, quominus inde poſſit determinatio curvæ deduci.

§. 219. *Problema.* Magnitudine detur perimeter $AMM'M''M'''\ldots$; ſed centrum gravitatis ejus reſpectu alicujus rectæ axi BB' perpendicularis debeat eſſe omnium maxime depreſſum. Fig. 61.

Eſt ideo $z + z' + z'' + z''' + z^{IV} + \ldots$ = dato
$z(b+c) + z'(c+c') + z''(c'+c'') + z'''(c''+c''') + \ldots$ = maximo.

$$\text{Hinc}\quad \frac{dz}{dv} + \frac{dz'}{dv} + \frac{dz''}{dv} + \frac{dz'''}{dv} + \frac{dz^{IV}}{dv} + \ldots = 0$$

$$(b+c)\frac{dz}{dv} + (c+c')\frac{dz'}{dv} + (c'+c'')\frac{dz''}{dv} + (c''+c''')\frac{dz'''}{dv} + \ldots = 0$$

dy

$$\begin{aligned}&\frac{dy}{dv}(\text{coſ.}PMA-\text{coſ.}P'M'M)\\&+\frac{dy'}{dv}(\text{coſ.}P'M'M-\text{coſ.}P''M''M')\\&+\frac{dy''}{dv}(\text{coſ.}P''M''M'-\text{coſ.}P'''M'''M'')=0;\\&+\frac{dy'''}{dv}(\text{coſ.}P'''M'''M''-\text{coſ.}P^{iv}M^{iv}M''')\\&+\ -\ -\ -\ -\ -\end{aligned}$$

$$\begin{aligned}&\frac{dy}{dv}((b+c)\,\text{coſ.}PMA-(c+c')\text{coſ.}P'M'M)\\&+\frac{dy'}{dv}((c+c')\text{coſ.}P'M'M-(c'+c'')\text{coſ.}P''M''M')\\&+\frac{dy''}{dv}((c'+c'')\text{coſ.}P''M''M'-(c''+c''')\text{coſ.}P'''M'''M'')=0.\\&+\frac{dy'''}{dv}((c''+c''')\text{coſ.}P'''M'''M''-(c'''+c^{iv})\text{coſ.}P^{iv}M^{iv}M''')\\&+\ -\ -\ -\ -\ -\ -\ -\end{aligned}$$

Omnes quantitates mutabiles $y,\ y',\ y'',\ y''',\ y^{iv}\ldots$ fiant ſimul conſtantes, duabus exceptis: erit pro vertice quolibet M, duos inter 'M, M' ſito, $\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{('x+x)\text{coſ.PM'M}-(x+x')\text{coſ.P'M'M}}$ quantitas conſtans ab Δx pendens; ſeu

$$\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{2x(\text{coſ.PM'M}-\text{coſ.P'M'M})-\Delta x(\text{coſ.PM'M}+\text{coſ.P'M'M})}$$ eſt quantitas conſtans:

proinde $$\frac{\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{\Delta x}}{2x\frac{\text{coſ.PM'M}-\text{coſ.P'M'M}}{\Delta x}-(\text{coſ.PM'M}+\text{coſ.P'M'M})}=\frac{1}{2C}$$

unde & limes hujus quantitatis, nempe

$$\frac{\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}}{x\frac{-\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}-\frac{dy}{dz}}=\frac{1}{C}.\quad\text{Hinc}\ \frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}=\left(x\frac{\frac{ddy}{dx^2}}{\left(\frac{dz}{dx}\right)^3}+\frac{dy}{dz}\right)\frac{1}{C}.$$

$$\frac{dy}{dz}=\frac{x\frac{dy}{dz}-C'}{C}$$

$$\frac{dy}{dz}=\frac{C'}{x-C}=\text{coſ.}PMT$$

$$\frac{dy}{dx}=\frac{C'}{\surd((x-C)^2-C'C')}.$$

Hoc

Hoc exemplo ad curvam tranſlato, eſt
$$\frac{dZ}{dx} = \frac{dz}{dx}$$
$$\frac{dZ'}{dx} = x\frac{dz}{dx}:$$

et poſita Z magnitudine data, Z' fit omnium maxima, quando $\frac{dy}{dz} = \frac{C'}{x-C'}$.

Exemplum hoc refertur ad curvam *catenariam* dictam.

§. 220. Exempla hactenus propoſita ſufficiunt, ut tirones videant, quomodo quæſtiones illis ſimiles, quæ evolutæ fuerunt, ad principia limitum poſſint reduci; & ut intelligant, methodos a mathematicis huic quæſtionum generi ſolvendo applicatas ſpectari debere tanquam compendia, ſolutioni brevius conſequendæ apta. Quoniam autem methodi hæ admodum ſunt generales, & iis quoque functionibus applicantur, quæ exponentes differentiales altiorum ordinum involvunt; quædam adhuc exempla facillima explicabo, quæ ſecundi altiorumve ordinum exponentes attingunt.

Sit $\frac{dZ}{dx} = \left(\frac{ddy}{dx^2}\right)^n$; & quæratur functio Z talis, ut ſit omnium maxima.

Quoniam $\frac{ddy}{dx^2} = \lim.\frac{\Delta^2 y}{\Delta x^2}$;

$$\begin{aligned}
&\text{fiat } Z' = ({}'y-2y+y')^n && \text{Erit } \frac{dZ'}{dv} = n({}'y-2y+y')\left(\frac{d'y}{dv}-2\frac{dy}{dv}+\frac{dy'}{dv}\right)\\
&\quad + (y-2y'+y'')^n && \quad + n(y-2y'+y'')\left(\frac{dy}{dv}-2\frac{dy'}{dv}+\frac{dy''}{dv}\right)\\
&\quad + (y'-2y''+y''')^n && \quad + n(y'-2y''+y''')\left(\frac{dy'}{dv}-2\frac{dy''}{dv}+\frac{dy'''}{dv}\right) \quad = o\\
&\quad + (y''-2y'''+y^{\text{iv}})^n && \quad + n(y''-2y'''+y^{\text{iv}})\left(\frac{dy''}{dv}-2\frac{dy'''}{dv}+\frac{dy^{\text{iv}}}{dv}\right)\\
&\quad + \;-\;-\;- && \quad + \;-\;-\;-\;-\;-\;-\\
&\quad + \;-\;-\;- && \quad + \;-\;-\;-\;-\;-\;-
\end{aligned}$$

Hinc omnibus quantitatibus mutabilibus ${}'y, y, y', y'', y''', y^{\text{iv}}\ldots$ factis ſimul conſtantibus, una excepta; fit pro quantitate quacunque mutabili y, inter quantitates ${}'y$, y' media, $({}'y-2y+y')^{n-1} - 2(y-2y'+y'')^{n-1} + (y'-2y''+y''')^{n-1} = o$.

Fiant

Fiant $({}'y - 2y + y')^{n-1} = {}'Q$

$(y - 2y' + y'')^{n-1} = Q$: erit $\Delta''Q = 0$; unde $\frac{\Delta''Q}{\Delta x^2} = 0$, & quantitatis hujus

$(y' - 2y'' + y''')^{n-1} = Q'$

limes, nempe $\frac{ddQ}{dx^2} = 0$: unde $\frac{dQ}{dx} = C^{N-1}$; $Q = C'^{N-1} + C^{N-1}x = \left(\frac{ddy}{dx^2}\right)^{n-1}$, &

$\frac{ddy}{dx^2} = (C'^{N} + C^{N-1}x)^{\frac{1}{n-1}}$; $\frac{dy}{dx} = c + \frac{n-1}{n}(C'^{N} + C^{N-1}x)^{\frac{n}{n-1}}$

$$y = c' + cx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1}(C'^{N} + C^{N-1}x)^{\frac{2n-1}{n-1}}.$$

Proinde poſito $\frac{dZ}{dx} = \left(\frac{ddy}{dx^2}\right)^n$, fit Z omnium maximum, ſi fuerit

$$y = c' + cx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1}(C'^{N} + {}^{N-1}Cx)^{\frac{2n-1}{n-1}}.$$

Eodem modo, ſi ſit $\frac{dZ}{dx} = \left(\frac{d^3y}{dx^3}\right)^n$: erit

$$y = c'' + c'x + \tfrac{1}{2}cxx + \frac{n-1}{n} \cdot \frac{n-1}{2n-1} \cdot \frac{n-1}{3n-1}(C''^{N} + C'^{N-1}x + \tfrac{1}{2}C''^{N-11}x^2)^{\frac{3n-1}{n-1}}.$$

Unde lex manifeſta eſt pro quolibet exponente differentiali.

Caput vicesimum primum.

Delineatio ſuccincta applicationis calculi differentialis et integralis ad phyſicam.

§. 221.

Hucusque nonniſi objecta ad matheſin puram ſpectantia tractavi. Quare ab rigore demonſtrandi, quem ea admittit atque exigit, deflectere non debui. Speciatim non licuit figuras curvilineas conſiderare tanquam rectilineas, quicunque harum ſit numerus laterum; nec ſolida ſuperficiebus curvis terminata tanquam polyhedra, utcunque magno hedrarum numero contineantur. Univerſim ab magnitudinibus limitum capacibus ad ipſos hos limites transire non licuit, niſi quoad tranſitus hic propoſitionibus eum firmantibus muniebatur; cujusmodi tum paſſim, tum Capite præſertim primo ſtabilire annixus ſum.

§. 222. Rigorem hunc in matheſi pura omnino neceſſarium, atque unum ex propriis ipſius characteribus conſtituentem, applicationes ejus ad objecta, quorum cognitio ab ſenſuum teſtimonio pendet (ſive ad artes pertineant, ſive ad ſcientias), non ſolum non exigunt; ſed & affectationem illius ut ſupervacaneam, quin ſæpe noxiam reſpuunt. (*a*) Certitudo mathematica ab conſtitutione organorum noſtrorum independens debet eſſe abſoluta: cognitioni noſtræ phyſicæ, ab indole ſenſuum noſtrorum pendenti, ſemper aliquid labis adhærebit ab imperfectione organorum noſtrorum, utcunque exercitatorum, & quacunque arte adjutorum. Translationem abſtractionum mathematicarum ad ſcientias phyſicas harum progreſſui officere poſſe experientia teſtatur: quod comprobantium exemplorum pauca exſtantiora recenſebo.

Quoad objectorum tantum terreſtrium & lunæ diſtantiæ ad indagandam velocitatem lucis ſuppetebant; determinari haud potuit tempus, quo lux datum ſpatium emetitur. Unde judicio præcipiti illatum fuit, lucem ſpatio cuicunque emetiendo nullum impendere tempus; velocitas ejus pronunciata fuit infinita: explicationes facti hujus nullo vellicati dubio traditæ fuerunt; quibus etiamnum adſtipulaturi eſſemus, niſi obſervationes in corporibus longe remotioribus inſtitutæ falſum eſſe, quod explicandum ſumebatur, docuiſſent.

Ex apparente directionum gravium parum invicem diſtantium parallelismo ſuperficies telluris diu plana fuit judicata: ad rotunditatem terræ ſtabiliendam alius generis obſervationibus opus fuit, quæ philoſophorum oculos ſerius demum perculerunt.

Ex ſenſibili ponderum corporum ad diſtantias parum diverſas a ſuperficie telluris æqualitate diu inferri conſuevit æqualitas actionis cauſæ gravitatis: donec phyſica exactior teſtimonii ſenſuum æſtimatione accuratiori & diſcuſſione phænomenorum immediate illis haud obviorum opinionem hanc delevit.

Alia

(*a*) Animadverſiones in applicationem matheſeos ad phyſicam Capite hoc traditæ, pariter ac deductiones expoſitæ Capite XII. Diſſertationis meæ: *Expoſition des principes des calculs ſuperieurs*, conformes ſunt doctrinis, quas ex ore Dni. Le Sage hauſi, quo tempore ſtudia mea philoſophica & mathematica affectu paterno regebat. Vid. Diſſ. cit. Nota p. 210.

Alia afferre poſſem exempla errorum, in quos præcipites ab phænomenis ad cauſas ipſorum concluſiones induxerunt; & qui ab ſimili judicii præcipitantia cavere monent.

§. 223. Per totum hunc librum oſtenſa infinite parvorum, quæ vocantur, mathematicorum inutilitate; & quibusdam locis, repugnantia conceptus ipſorum: diverſam ab ea, quod ad objecta phyſica, ſententiam cum Dno. LE SAGE profiteor. In ſcientiis ad hæc applicatis pro nihilo habemus, quod ſenſus noſtros quacunque adjutos arte effugit: & explicatio phænomenorum naturæ, conformitatem eorum non rigoroſam quidem cum legibus arbitrariis inde deductis, ſed eousque accuratam, quousque obſervationum noſtrarum ἀκρίβεια pertingit, oſtendens, hoc reſpectu ſatisfacere propoſito cenſenda eſt. (*b*)

§. 224. Principio hoc admiſſo, Dn. LE SAGE omnino convellit objectiones mechanicæ phænomeni naturæ latiſſime patentis, gravitatis univerſalis, explicationi adverſas; adhibitis (juxta analyſin æque ingenioſam ac ſolidam) *corpuſcu-*

(*b*) Quo tempore infinite parva tanquam baſis neceſſaria calculorum ſuperiorum ſpectabantur: Dn. LE SAGE uſum eorum in matheſi pura ab contradictionibus, quas implicat, tueri ſatagebat conceptu ſequenti: quem deinde ad eam nonniſi ſenſu indirecto vel devio & explicatu prolixo applicari poſſe agnovit; ac quem ipſiusmet verbis, 14 Julii 1786. ad me perſcriptis, uti in Diſſertatione mea priore pag. 211. exponam:

„Mon but dans cette ancienne recherche etoit d'aſſigner une bonne fois aux infi„niment petits de tout genre & de tout ordre une conſtitution, qui me permit en„ſuite & toujours de les traiter hardiment comme des quantités determinées, finies, „mais negligibles; & qui fut applicable aux etres réels, comme aux etres hypothe„tiques.

„Pour cet effet je ſubſtituai aux quantités moindres qu'aucune aſſignée, & à cel„les qu'on pourroit pretendre etre moindres qu'aucune aſſignable, des quantités moin„dres qu'aucune, qu'on aſſignera réellement jamais; ou plutot des parcelles, qu'on „peut negliger relativement à leur tout, ſans aucune erreur qui devienne jamais réel„lement ſenſible.

„Effectivement de telles quantités ſont vraiment *determinées*; puisque toute ap„plication, qu'on fera réellement jamais, de quelque propoſition que ce ſoit, eſt de„terminée en ſoi, quoique actuellement inconnue: vu que tous les futurs contingens „ſont determinés, n'y eut-il meme aucun etre qui les previt, ni aucun *nexus* entre „ces futurs-là & le preſent.

puſculis, quæ intervallis relate ad dimenſiones ipſorum admodum magnis invicem diſtent, & quaquaverſus (*c*) *ſecundum lineas rectas celeritate prægrandi moveantur.*

Dn. Le Sage actionis fluidi hujus conſequentias mathematicas accurate evolvit, & conſenſum earum cum phænomenis gravitatis oſtendit. Meditationes ipſius de arduo hoc phyſicæ generalis objecto minus adhuc notæ ſunt. Studio, qua poſſunt, perfectione redactas publico exponendi, ſcripta ſua divulgare non feſtinavit: nec niſi primas ſyſtematis ſui lineas in Diſſertatione inſcripta *Eſſai de chymie mecanique*, quam Academia Rhotomagenſis anno 1758. præmio ornavit, ac nuper in *Lucretio Newtoniano*, Commentariis Academiæ Berolinenſis anni 1782. inſerto, expoſuit. Ambitus ejus opera nonnullorum ipſius amicorum quodammodo innotuit: quos inter Dni. De Luc in variis ſuis ſcriptis, iis nominatim, quæ ſub titulo: *Idées ſur la Meteorologie*, *Lettres à la Reine d'Angleterre*, *Lettres à Mr. de la Metherie*, edidit; & Dni. Prevost in libro inſcripto *Origine des forces magnetiques.* (*d*) Grato in agnatum hunc pie devenerandum, cujus curæ quicquid in me eſt doctrinæ debeo, affectu motus, cum ſeptennio abhinc ex Polonia reverterer, ſummopere optavi exiguum ejus documentum ipſi exhibere opera mea qualicunque ad manuſcriptorum ipſius multiplicium editionem promovendam. Quo jucundior mihi fuit, quam diu animo fovi, ſpes hoc voto potiundi, eo magis irritam ab me conceptam fuiſſe dolui.

(*c*) Hæc quoque expreſſio phyſice eſt intelligenda. Quod innuo difficultatis ſolvendæ gratia, nuper ab illuſtri Mathematico motæ, & uberius ab Dno. Wilckens expoſitæ, contra pag. 71 ſq. Diſſertationis: *Eſſai de chymie mecanique*; ubi diſpoſitiones quadratiformes punctorum in ſuperficie ſphæræ rigoroſo ſenſu mathematico accipi non debent. Qui primus nævum doctrinæ Elementorum Euclidis de angulis ſolidis animadvertit geometra (*Hiſt. de l'Acad. royal. des Sc.* 1756. *Bermanni Commentatio de angulis ſolidis*); poteratne in gravem incidere errorem, quem commiſiſſet, ſi expreſſiones ipſius ad diſcuſſionem phyſicam pertinentes mathematice eſſent interpretandæ? (Vid. *Wilckens Auffäze mathemat. Inhalts.* Gött. 1790. *Käſtners geometr. Abhandl.* Gött. 1791.)

(*d*) Dn. Le Sage non ſolum arduum gravitatis univerſalis phænomenon per actionem immediatam corpuſculorum ſupra indicatorum explicat: ſed eadem quoque mediatam plerorumque phænomenorum naturæ generalium cauſam eſſe oſtendit, quatenus activitatem ſuam ipſis debeant fluida elaſtica, per quæ phænomena illa poſſunt explicari.

§. 225. Conceptus actionis discontinuæ fluidi gravifici, phyſico ſua ſe luculentia commendans, ſæpe etiam mathematico commodus eſt. Theoriæ fluidorum continuorum, elaſticorum pariter ac non elaſticorum, admodum imperfectæ adhuc ſunt, quamvis ſummi hujus & ſuperioris ſeculi mathematici ſua illis ſtudia impenderint; & quas a priori ſtabilire quidam aggreſſi ſunt, experientia evertit. Conſequentias principiorum, quæ pro baſibus ſumuntur, perſequi exercitatiſſimis tantum calculatoribus datur. Alia eſt ratio theoriæ fluidorum diſcretorum. Principia ejus ſunt indubitata; & conſequentiæ deductu faciles; quod calculi Dni. Le Sage comprobant, qui ne primas quidem calculorum ſuperiorum notiones exigunt.

§. 226. Ex. gr. Galilei lex accelerationis gravium (phyſice intellecta) immediate ex actione fluidi ejusmodi conſequitur. Cum incrementum velocitatis gravium debeatur fluido, tempuſculis æqualibus per impulſus æquales agenti: celeritas, quam gravia tempore ſenſibili acquirunt, numero harum impulſionum; proinde tempori proportionalis eſt, quo corpora illas ſunt experta. Immedite igitur conſequitur, eſſe $v \backsim t$.

Hinc ſpatia tempuſculis æqualibus ſucceſſivis percurſa, utpote celeritatibus acquiſitis proportionalia, pariter creſcunt uti tempora, proinde uti numeri naturales; & ſpatia inde ab initio motus percurſa creſcunt uti numeri triangulares numerorum momentorum illorum temporis. Sed numerorum triangularium ratio eo propius accedit ad rationem numerorum quadratorum correſpondentium, quo majores ſunt numeri, qui ipſorum ſunt radices (§. 20.). Itaque ſpatia percurſa eo accuratius proportionalia ſunt quadratis temporum elapſorum, quo plura quodvis tempus ſenſibile continet tempuſcula, duobus cauſæ motricis impulſibus immediate ſeſe conſequentibus interjecta.

§. 227. Optabile ſine dubio foret, ut mathematicis ſemper liceret eadem facilitate conſequentias principii aſſumti deducere. Magis autem aſſueti exponentibus differentialibus $\frac{ds}{dt} = v$, $\frac{dv}{dt} = p$, aliisque ſimilibus; quam expreſſionibus per differentias finitas, quæ illis reſpondent: priores tractare expreſſiones, quam poſteriores, magis fere (in inveſtigationibus præſertim prolixis ac difficili-

liori-

lioribus) commodum ducunt; ideoque eas alteris ſubſtituunt, quibus propoſitum facilius conſequuntur. Qua quidem ſubſtitutione uti ipſis licet: dummodo rationis, qua ducti eam uſurpant, non obliviſcantur; neque exigant, ut, quas inde nectunt, conſequentiæ abſoluto ac mathematico rigore veræ eſſe cenſeantur.

Ex æquatione $v = t$, vel potius $v = gt$ (poſita g celeritate, quam corpus acquirit tempore pro unitate ſumto, ex. gr. uno minuto ſecundo) conſequitur $\frac{dv}{dt} = g$.

Cum autem ſit $\frac{ds}{dt} = v$; ſit $\frac{d^2 s}{dt^2} = \frac{dv}{dt} = g$.

Porro cum ſit $\frac{dv}{dt} = g$;

& $v = \frac{ds}{dt}$:

ſequitur $v\frac{dv}{dt} = g\frac{ds}{dt}$; & $v\frac{dv}{ds} = g$.

Hinc, ob g conſtantem, ſit $\frac{1}{2}v^2 = gs$; $v = \sqrt{2gs}$.

Sed $v = gt$. Ergo $gt = \sqrt{2gs}$; & $t = \sqrt{\frac{2s}{g}}$.

§. 228. Formula differentialis $\frac{dv}{dt} = g$ pariter applicatur ad caſum, quo g variabilis eſt; & qui, phyſice intellectus, ſupponit tempus lapſus in partes adeo exiguas dividi, ut, durante qualibet earum, actio cauſæ corpus impellentis ſit uniformis.

Exemplum primum. Gravitas variet uti diſtantia a centro.

Sit a diſtantia ab centro, a qua inde corpus delabitur;

g actio gravitatis ad hanc diſtantiam.

Corpore delapſo per ſpatium s, gravitas ad diſtantiam $a - s$ erit $g \times \frac{a-s}{a}$.

Igitur $\frac{dv}{dt} = g \times \frac{a-s}{a}$

ſed $\frac{dt}{ds} = \frac{1}{v}$

Ergo $v\frac{dv}{ds} = g \times \frac{a-s}{a}$

$v^2 = C + g \times \frac{2as - s^2}{a}$

 Sed

Sed v evaneſcit, quando $s = 0$. Proinde $C = 0$; $v^2 = g \times \frac{2as - s^2}{a}$.

Sit $s = a$; erit $v^2 = ag$. Sed gravitate poſita uniformi, erat $v^2 = 2ag$ (§. 227.) Quadratum igitur celeritatis acquiſitæ per actionem gravitatis conſtantis duplum eſt quadrati celeritatis acquiſitæ per actionem gravitatis variabilis juxta legem propoſitam.

Porro $\frac{dt}{ds} = \frac{1}{v} = \sqrt{\frac{a}{g}} \times \frac{1}{\sqrt{(2as - s^2)}}$; unde $t = \sqrt{\frac{a}{g}}\left(C - \text{arc.ſin.}\frac{a-s}{a}\right)$

Sed $t = 0$, dum $s = 0$. Ergo $C = \text{arc.ſin.}1 = p$;

$$t = \sqrt{\frac{a}{g}}\left(p - \text{arc.ſin.}\frac{a-s}{a}\right) = \sqrt{\frac{a}{g}} \times \text{arc.coſin.}\frac{a-s}{a}.$$

Sit $s = a$; erit $t = \sqrt{\frac{a}{g}} \times p$. Et cum ſit $g \backsim a$: eſt $t \backsim p$; ſeu tempus lapſus ad centrum usque conſtans eſt, quodcunque ſit ſpatium percurrendum: quod principium eſt iſochronismi in cycloide.

Exemplum ſecundum. Gravitas ſit in ratione inverſa quadrati diſtantiæ.

Erit $\frac{dv}{dt} = g \times \frac{a^2}{(a-s)^2}$

$\frac{dt}{ds} = \frac{1}{v}$

Ideoque $v\frac{dv}{ds} = g \times \frac{a^2}{(a-s)^2}$; $\frac{1}{2}v^2 = C + g \times \frac{a^2}{a-s}$.

Sit $v = 0$, dum $s = 0$: erit $C = -g$; $\frac{1}{2}v^2 = g\left(\frac{a^2}{a-s} - a\right) = g \times \frac{as}{a-s}$

$$v = \sqrt{2ag} \times \sqrt{\frac{s}{a-s}} = \sqrt{\frac{2}{a}} \times \sqrt{\frac{s}{a-s}} \text{ ob } g = \frac{1}{a^2}.$$

Porro $\frac{dt}{ds} = \frac{1}{v} = \frac{1}{\sqrt{2ag}} \times \sqrt{\frac{a-s}{s}} = \frac{1}{\sqrt{2ag}} \times \frac{a-s}{\sqrt{(as-s^2)}} = \frac{1}{\sqrt{2ag}}\left(\frac{\frac{1}{2}a}{\sqrt{(as-s^2)}} + \frac{\frac{1}{2}a - s}{\sqrt{(as-s^2)}}\right)$

Unde $t = \frac{1}{\sqrt{2ag}}\left(\frac{1}{2}a \times \text{arc.ſin.verſ.}\frac{s}{\frac{1}{2}a} + \sqrt{(as - s^2)} + C\right)$

Sit $t = 0$, dum $s = 0$: erit $C = 0$; $t = \frac{1}{\sqrt{2ag}}\left(\frac{1}{2}a \times \text{arc.ſin.verſ.}\frac{s}{\frac{1}{2}a} + \sqrt{s(a-s)}\right)$.

Sit $s = a$. Erit $t = \frac{1}{\sqrt{2ag}} \times \frac{1}{2}a\pi = \frac{1}{\sqrt{2ag}} \times ap$.

Atqui $g = \frac{1}{a^2}$. Ergo $t = \frac{a\sqrt{a}}{\sqrt{2}}p$; & $t^2 = \frac{a^3}{2}p^2$.

Obſer-

Obſervatio. Cum expreſſiones temporis t & celeritatis v imaginariæ fiant pro $s > a$; docemur, motum ultra centrum gravitationis locum habere non poſſe.

Sit $s = a$: erit $v = \sqrt{\frac{2}{a}} \times \sqrt{\frac{a}{a-a}} = \sqrt{\frac{2}{a}} \times \sqrt{\frac{a}{0}}$; quod eſt ſignum impoſſibilitatis (Cap. IX.): ideoque corpus non ſolum ultra centrum gravitationis progredi nequit; ſed ne ad centrum quidem usque poteſt pervenire. Quod geometrice oſtendi poteſt modo ſequente.

Cum ſit $v^2 = \frac{2}{a} \times \frac{s}{a-s} = \frac{2}{a}\left(\frac{a}{a-s} - 1\right) = \frac{2}{a-s} - \frac{2}{a} = \frac{2aC^2}{a-s} - 2C^2$;

eſt $v^2 + 2C^2 = \frac{2aC^2}{a-s}$.

Deſcripta igitur curva hyperbolica, cujus centrum ſit centrum gravitationis, & cujus abſciſſæ ab centro inde in aſymptoto alterutra ſumtæ ſint ſpatia percurrenda, cujus denique æquatio ſit $y^2 + b^2 = \frac{2ab^2}{x}$; quadrata celeritatum, aucta quantitate data, proportionalia erunt quadratis ordinatarum: pariterque ac ordinatæ nullum habent limitem magnitudinis, nec celeritas limitem magnitudinis habet; atque uti nullum eſt hyperbolæ punctum, cui ordinata reſpondeat occurrens axi abſciſſarum in centro; ita nec celeritas ulla eſt, quæ reſpondeat ſuppoſitioni, quod corpus ad centrum usque poſſit pervenire: proinde ſuppoſitio hæc impoſſibilis eſt.

Caſus hic plures occupavit mathematicos; quorum nonnulli valde paradoxa de eo ſtatuerunt. Exemplum, mea quidem ſententia, is præbet abuſus abſtractionum mathematicarum, objectis applicatarum realibus, cum quibus conſiſtere nequeunt.

Gravitatio, quam corpus ſphæricum exercet, ſequitur rationem inverſam duplicatam diſtantiæ ab centro corporis hujus, quoad corpus gravitans extra illud ſitum eſt. Poſito autem, corpus gravitans poſſe (per fictionem aliquam admiſſibilem) intra illud pervenire: tum lex gravitationis eadem, quæ prius, non ſubſiſtit; ſed gravitatio variatur in ratione ſimplici diſtantiæ a centro. Juxta hanc vero legem corpus ad centrum usque delabitur, & velocitatem determi-

terminatam acquirit: tum ultra centrum progreditur motu retardato ſimili ratione, qua acceſſus ipſius ad centrum fuerat acceleratus. Conſiderationes igitur phyſicæ ſententiam hoc caſu confirmant, quam de ſignificatione ſymboli $\frac{1}{0}$ Capite IX. profeſſus ſum. Conf. Diſs. Dn. Le Sage inſerta Diario *Journal de phyſique de l'Abbé Rozier. Janvier* 1776.

Formulæ præcedentes applicantur ad motus juxta legem quamcunque datam variatos; nominatim ad motus corporum in mediis reſiſtentibus. Sed ſcopus præſentis ſcriptionis applicationes has perſequi prohibet. Quare ad motus compoſitos progredior.

§. 229. Kepleri celeberrima lex arearum inſigne præbet exemplum facilitatis, quam inveſtigationibus phyſico-mathematicis conciliat ſuppoſitio actionis non continuæ cauſæ motricis. Quippe juxta eam demonſtratio legis hujus nonniſi ſimpliciſſimas requirit geometriæ elementaris propoſitiones, eas nimirum, quibus nititur æqualitas triangulorum æquealtorum ſuper eadem baſi. Re etiam ipſa omnes fere auctores, præeunte Newtono, eam juxta hanc ſuppoſitionem explicant; quicquid ceterum de conformitate ſuppoſiti hujus cum natura rei ſentiant: & tum ad ſemitas curvilineas applicant, quod in rectilineis locum habet, utcunque exigua ſint latera ipſarum.

Demonſtratio curvis ipſis immediate applicata æque facilis non eſt. Comparationis duarum methodorum inſtituendæ gratia poſteriorem tradam eo fere modo, quo ipſam expoſuit cel. de la Place in *Theoria* ſua *motus plane'arum.*

Sit S focus gravitationis; SM radius vector, ſeu diſtantia, $= r$; M corpus gravitans; & gravitas ſit cuicunque diſtantiæ SM functioni $= \varphi r$ proportionalis.

Per focum S agantur duæ quæcunque rectæ SP, SQ, invicem normales; ad quas referatur motus corporis M. Sint $MP = SQ = x$, $MQ = y$.

Actio gravitatis ſecundum directionem MQ erit $\varphi r \times \frac{y}{r}$; ſecundum directionem SQ, $\varphi r \times \frac{x}{r}$: & celeritates planetæ juxta has directiones reſpective erunt $\frac{dy}{dt}$, $\frac{dx}{dt}$. Exponentes differentiales celeritatum harum, qui viribus $\varphi r \times \frac{y}{r}$, $\varphi r \times \frac{x}{r}$ proportionales ſunt, reſpective erunt $\frac{d^2y}{dt^2}$, $\frac{d^2x}{dt^2}$. Unde duæ conſequuntur

tur æquationes:

$$\frac{d^2y}{dt^2} = \varphi r \times \frac{y}{r}. \quad \text{Hinc} \quad x \times \frac{d^2y}{dt^2} = \varphi r \times \frac{xy}{r}$$

$$\frac{d^2x}{dt^2} = \varphi r \times \frac{x}{r} \qquad y \times \frac{d^2x}{dt^2} = \varphi r \times \frac{xy}{r}$$

$$\text{Ergo} \quad x \times \frac{d^2y}{dt^2} - y \times \frac{d^2x}{dt^2} = 0$$

$$\text{et} \quad x \times \frac{dy}{dt} - y \times \frac{dx}{dt} = C.$$

Sit angulus $MSQ = z$. Erunt $y = r\text{ſin}.z$; $\frac{dy}{dt} = r\,\text{coſ}.z\frac{dz}{dt}$

$$x = r\text{coſ}.z; \; \frac{dx}{dt} = -r\,\text{ſin}.z\frac{dz}{dt}$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\text{coſ}.^2z\frac{dz}{dt} + r^2\,\text{ſin}.^2z\frac{dz}{dt} = r^2\frac{dz}{dt} = C.$$

Poſita autem area curvæ, quam verrit radius vector, $= S$, eſt $\frac{dS}{dz} = \frac{1}{2}r^2$ (§.104.)

Quare etiam $\frac{dS}{dt} = \frac{1}{2}r^2\frac{dz}{dt} = C$

ideoque $\quad S = C' + Ct.$

Proinde ſi S evaneſcit, quando $t = 0$: fit $S = Ct$; ſeu area S eſt tempori proportionalis.

§. 230. Ex lege hac immediate duo corollaria ſequentia fluunt:

1°. Celeritas mobilis eſt in ratione inverſa perpendiculi, quod ab foco in directionem corporis demittitur; proinde in curva velocitas eſt in ratione inverſa producti ex radio vectore ac ſinu anguli, quem radius vector cum tangente curvæ comprehendit. Ideoque expreſſio velocitatis eſt $\frac{\sqrt{\left(rr + \left(\frac{dz}{dr}\right)^2\right)}}{rr}$ (§. 49.)

2°. Celeritas angularis mobilis, relata ad focum gravitationis, ſequitur rationem inverſam duplicatam radii vectoris.

§. 231. In gratiam phyſicorum, quibus calculi ſuperiores & theoria curvarum minus ſunt familiares, Dn. Le Sage conſequentias actionis haud continuæ cauſæ gravitationis & motus corporum in orbitis rectilineis perſequi ſategit; ac ſpeciatim in polygonis regularibus theoremata Hugenii ad vires centrales in circulo pertinentia demonſtravit.

Cum in omni curva vis centralis reſpondens tempori, quo mobile arcum aliquem curvæ percurrit, ſit in ratione compoſita ex directa ſimplice ſegmenti radii vectoris ad unum arcus hujus extremum ducti, quod arcui huic & tangenti per alterum arcus extremum ductæ interjacet, & ex inverſa ratione duplicata temporis ad arcum percurrendum impenſi: data curvæ, quam mobile deſcribit, æquatione ad focum gravitationis relata; ope formularum §§. 104. 196. determinatur lex gravitationis.

Fig. 57. Nempe poſita vi centrali $= g$; eſt $g = \lim. \frac{M't}{MFM'^2}$

Atqui $\lim. \frac{\Delta z}{MFM'} = \frac{1}{r^2}$, & $\lim. \frac{\Delta z^2}{MFM'^2} = \frac{1}{r^4}$. (§. 104.)

$$\text{ac} \quad \lim. \frac{M't}{\Delta z^2} = \tfrac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}$$

Igitur $$g\left(= \lim. \frac{M't}{MFM'^2}\right) = \frac{1}{r^4}\left(\tfrac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}\right).$$

Exemplum. Sit $r = \frac{bb}{a + e\,\mathrm{coſ.}z}$, quæ eſt æquatio focalis ellipſis;

itaque $$\frac{dr}{dz} = \frac{bbe\,\mathrm{ſin.}z}{(a + e\,\mathrm{coſ.}z)^2} = \frac{e\,\mathrm{ſin.}z}{bb} rr$$

$$\frac{ddr}{dz^2} = \frac{e\,\mathrm{coſ.}z}{bb} rr + \frac{2e\,\mathrm{ſin.}z}{bb} r \frac{dr}{dz} = \frac{e\,\mathrm{coſ.}z}{bb} rr + \frac{2ee\,\mathrm{ſin.}^2 z}{b^4} r^3$$

erit $$g = \frac{1}{r^4}\left(\tfrac{1}{2}r + \frac{ee\,\mathrm{ſin.}^2 z}{b^4} r^3 - \tfrac{1}{2} \times \frac{e\,\mathrm{coſ.}z}{b^2} r^2 - \frac{ee\,\mathrm{ſin.}^2 z}{b^4} r^3\right) = \tfrac{1}{2} \cdot \frac{1}{r^3}\left(1 - \frac{e\,\mathrm{coſ.}z}{bb} r\right)$$

Sed $er\,\mathrm{coſ.}z = bb - ar.$

Ergo $$g = \tfrac{1}{2} \times \frac{1}{r^3}\left(1 - \frac{bb - ar}{bb}\right) = \tfrac{1}{2} \times \frac{a}{bb} \times \frac{1}{rr}.$$

Pro-

Proinde in diverſis ejusdem ellipſis punctis gravitatio ſequitur rationem inverſam duplicatam diſtantiæ ab foco.

Eodem modo demonſtratur, eandem legem in ceteris ſectionibus conicis obtinere.

Viciſſim, data lege gravitationis, ex æquatione

$g = \frac{1}{r^4}\left(\frac{1}{2}r + \frac{1}{r}\left(\frac{dr}{dz}\right)^2 - \frac{1}{1.2} \times \frac{ddr}{dz^2}\right)$ deducitur natura curvæ deſcriptæ (quoad limites perfectionis calculi integralis concedunt).

Ex. gr. gravitate ſequente rationem inverſam duplicatam diſtantiæ, curva invenitur eſſe ſectio aliqua conica; quæ determinatur per angulum projectionis, & per rationem, quam celeritas projectionis habet ad eam, qua fieret, ut mobile circulum ad eandem diſtantiam deſcriberet.

Verum hæc fuſius deducere ab ſcopo libri hujus eſt alienum.

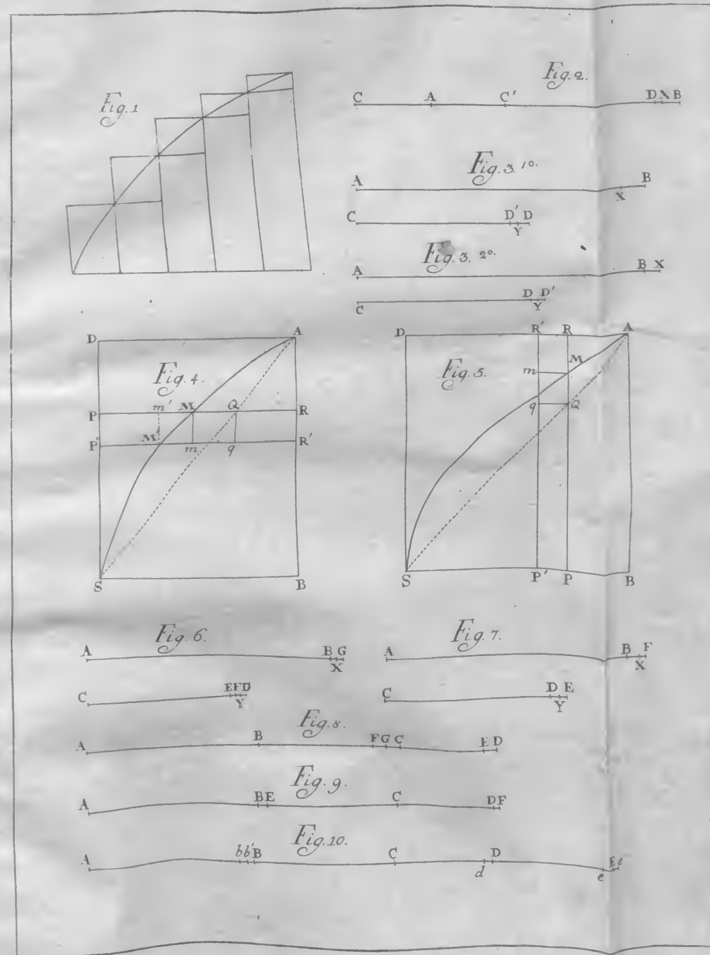
Corri-

Corrigenda.

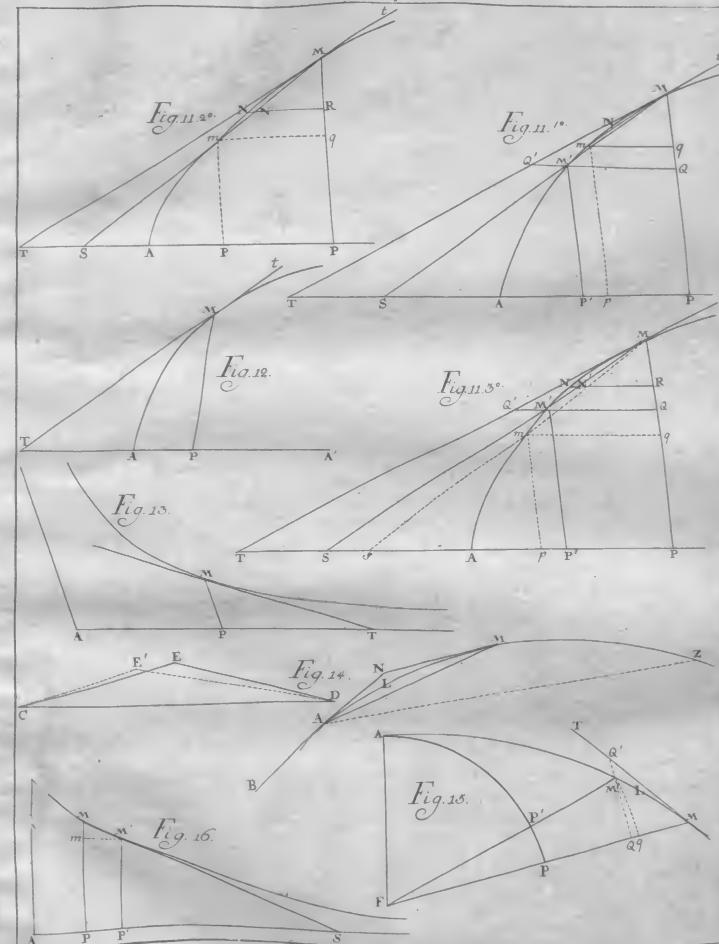
pag.	lin.	loco	legatur
27.	3.	terminorum usque	terminorum poſt primum usque
ibid.	3 ſq.	terminum, cujus coëfficiens L.	terminum, qui præcedit eum, cujus coëfficiens eſt L, una cum hujus duplo.
28.	1.	n^n	x^n
ibid.	2.	$\frac{n-(m+r)}{m+r+1}$	$\frac{n-(m+r-1)}{m+r}$
29.	18.	x	y
33.	22.23.	n	$n-1$
52.	15.	P''	P^N
73.	ult.	$>$	$<$
74.	1.3.4.	$>$	$<$
ibid.	2.	$<$	$>$
ibid.	ibid.	tangentis creſcentis	tangentis decreſcentis
77.	15.	$x=C+\frac{m}{n}$	$x=C+\frac{m}{n}y$
81.	9.	TS	MT
96.	22.	$\frac{1}{5}v^6$	$\frac{1}{6}v^6$
108.	18.	(§.)	(§. 1. *Introd.*)
113.	3.	$(z+\Delta)z$	$(z+\Delta z)$
118.	15.	2. 2	3. 2
163.	18.	b' & b	b' & b''
167.	ult.	$\sqrt{((aa+ee)^2-4ae\,\text{ſin.}^2 x)}$	$\sqrt{((a+e)^2-4ae\,\text{ſin.}^2 \tfrac{1}{2}x)}$
174.	25.	$2MP\times Mm'$	$2MP+Mm'$
189.	6.	$B,\ B'$	$A',\ B'$
190.	2.	$=$	$+$

ACCADEMIA
R. DELLE SCIENZE
DI TORINO

Tab. I.


Fig. 1
Fig. 2
C A C' D X B
Fig. 3. 1°.
A X B
C D' D Y
Fig. 3. 2°.
A B X
C D D' Y
Fig. 4.
D A P R P' R' m' M Q M' m q S B
Fig. 5.
D R' R A m M q Q S P' P B
Fig. 6.
A B G X
C E F D Y
Fig. 7.
A B F X
C D E Y
Fig. 8.
A B F G C E D
Fig. 9.
A B E C D F
Fig. 10.
A b b' B C D d E e

Tab. II.



Tab. III.

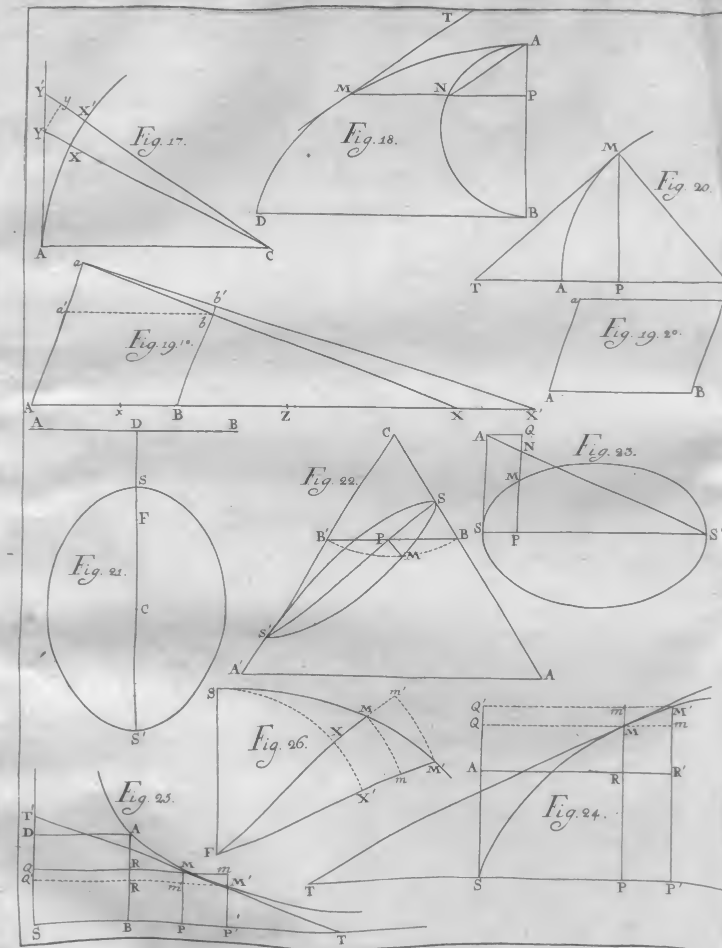
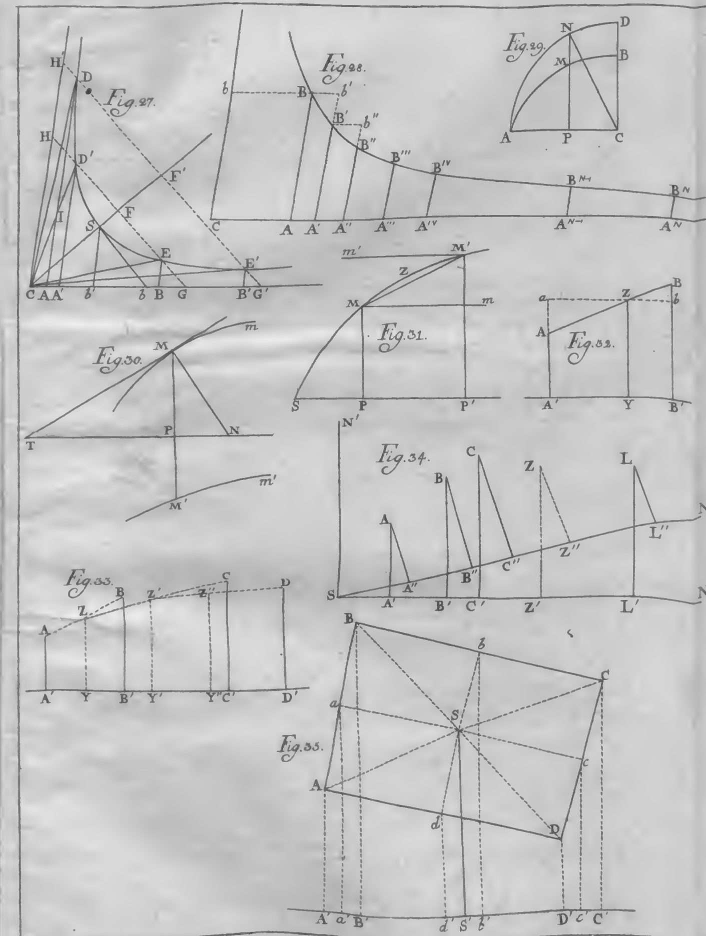
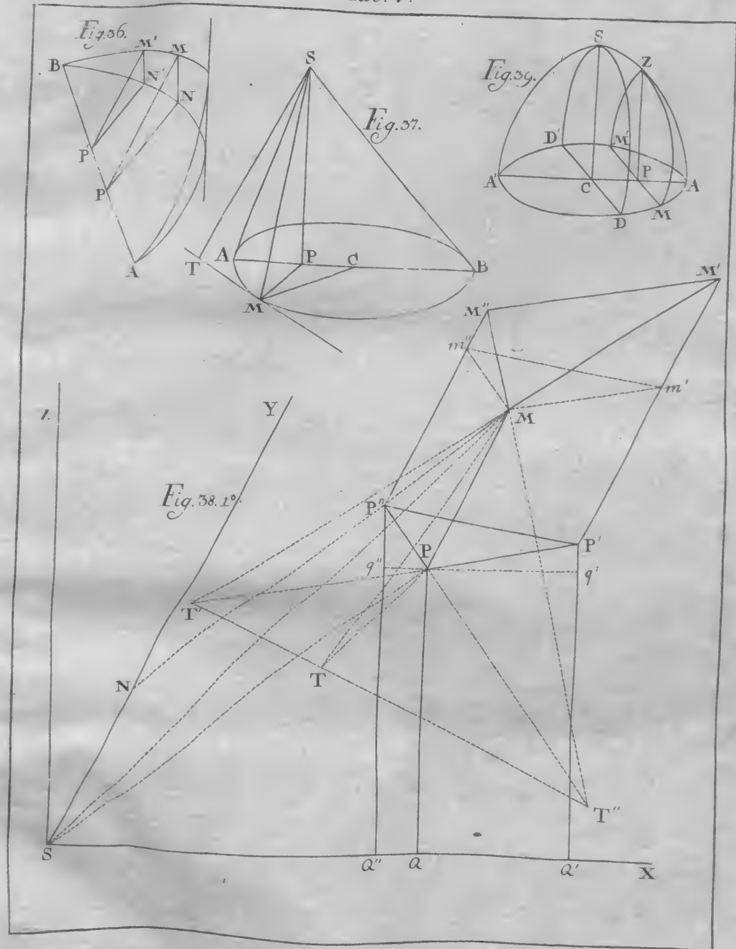


Fig. 17.
Fig. 18.
Fig. 20.
Fig. 19. 1°.
Fig. 19. 2°.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 26.
Fig. 25.
Fig. 24.

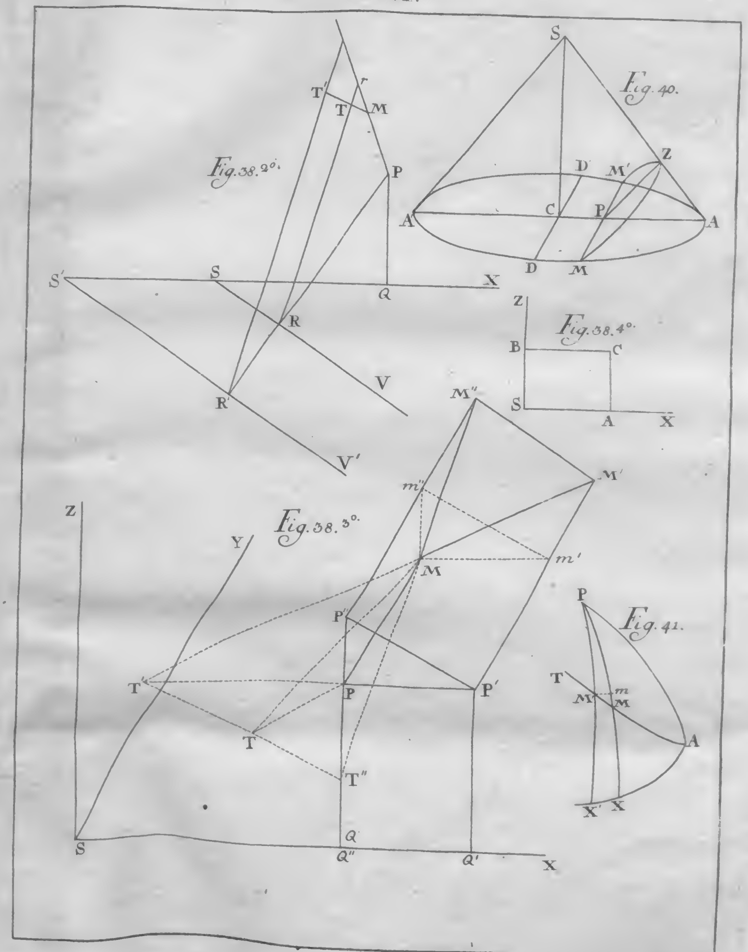
Tab. IV.



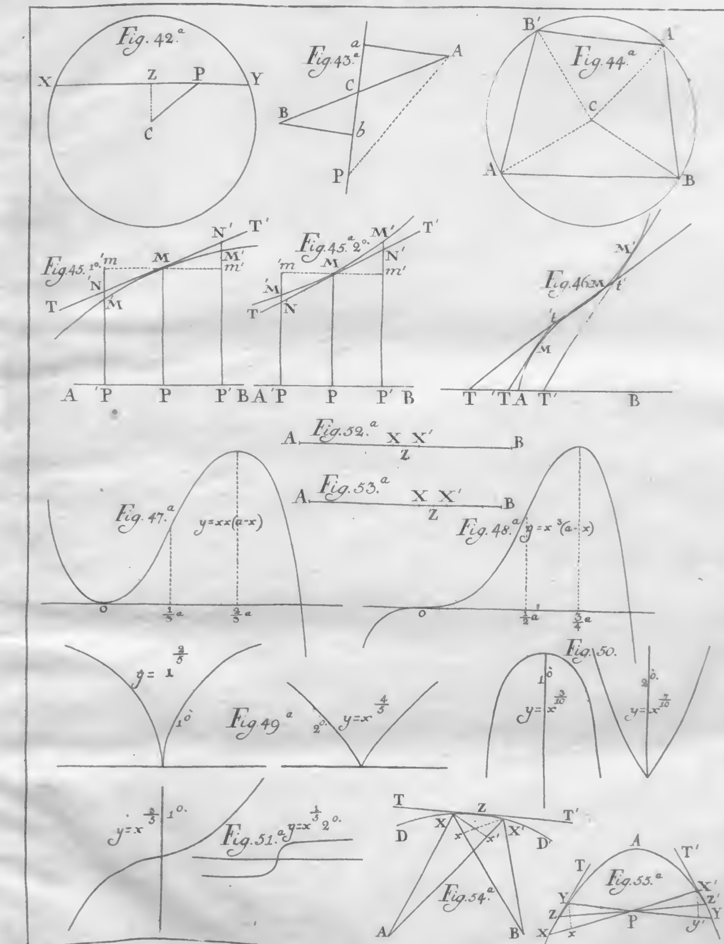
Tab. V.ª


Fig. 36.
B
M'
M
N'
N
P'
P
A
Fig. 37.
S
T
A
P
C
B
M
Fig. 39.
S
Z
D'
M'
A'
C
P
A
D
M
Fig. 38. 1°.
Z
Y
M'
M''
m''
m'
M
P''
P
P'
q''
q'
T'
T
N
T''
S
Q''
Q
Q'
X

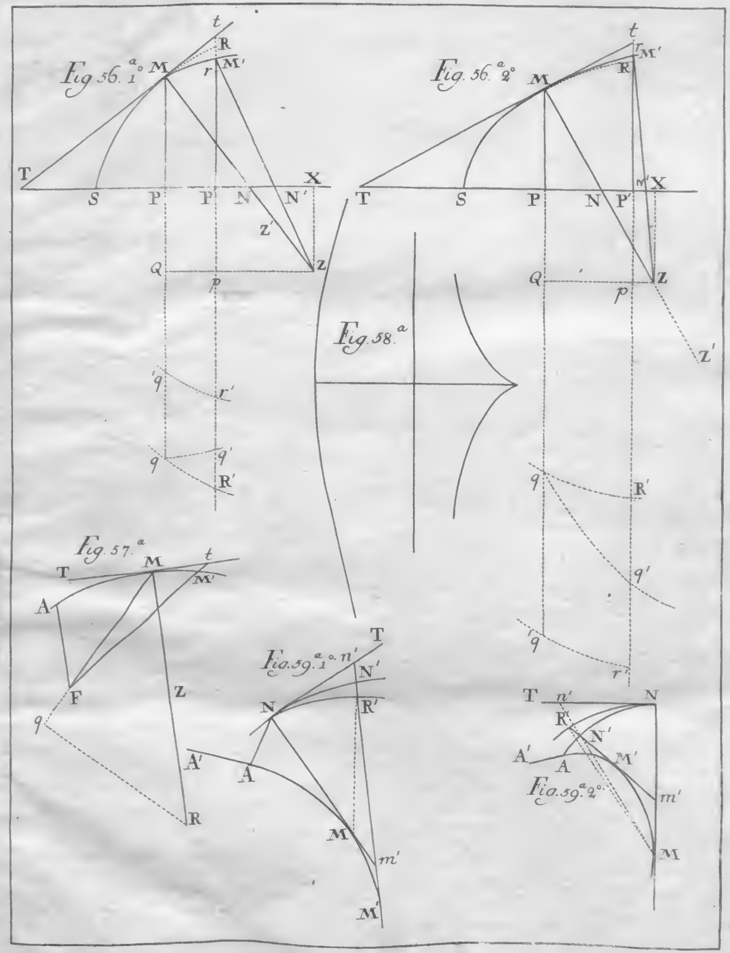
Tab. VI.ª



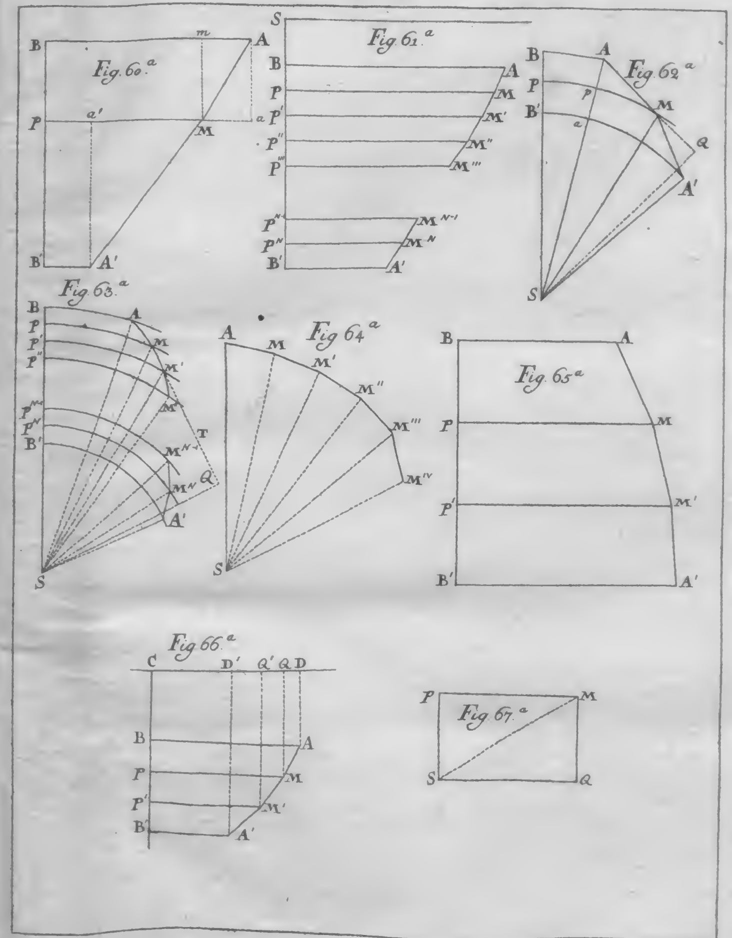
Tab. VII.ma


Fig. 42.a
Fig. 43.a
Fig. 44.a
Fig. 45. 1°.
Fig. 45.a 2°.
Fig. 46.
Fig. 52.a
Fig. 53.a
Fig. 47.a
$y = xx(a-x)$
Fig. 48.a
$y = x^3(a-x)$
$y = x^{\frac{2}{5}}$
Fig. 49 a
$y = x^{\frac{4}{5}}$
Fig. 50.
$y = x^{\frac{3}{10}}$
$y = x^{\frac{7}{10}}$
$y = x^{\frac{2}{5}}$
Fig. 51.a
$y = x^{\frac{1}{3}}$
Fig. 54.a
Fig. 55.a

Tab. VIII.ª



Tab. IX.ª



K II. 800